Integrator Forwarding

Daniele Carnevale

Dipartimento di Ing. Civile ed Ing. Informatica (DICII),
University of Rome “Tor Vergata”

References:

- D Carnevale, A Astolfi, “Integrator forwarding without PDEs”, CDC, 2009
Consider the system in **strict-feedforward form** (*upper-triangular structure*)

\[
\dot{\eta} = h(x), \\
\dot{x} = f(x) + g(x)u,
\]

with \(\eta \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathbb{R}^p\), where \(f(\cdot), h(\cdot) \in C^q\) with \(q \in \mathbb{N}_{\geq 1}\) are zero at zero, \(g(\cdot)\) is continuous and \(g(0) \neq 0\).

**Assumption (Stabilizability − x = 0 GAS+LES)**

The Jacobian linearization of (1)-(2) at \((\eta, x) = (0, 0)\) is stabilizable (controllable). The origin of the system (2), with \(u \equiv 0\), is GAS and LES with Lyapunov function \(V(x)\).

\[
\downarrow
\]

There exists a class-\(\mathcal{K}\) function \(\rho(\cdot)\) that is locally quadratic around the origin and such that

\[
\dot{V} = \frac{\partial V(x)}{\partial x} f(x) \leq -\rho(\|x\|), \text{ with } u(t) \equiv 0.
\]
Assumption 1 and the properties of $h(x)$, with $u \equiv 0$, are sufficient conditions for the existence of a map $M(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$h(x) = \frac{\partial M(x)}{\partial x} f(x), \quad M(0) = 0. \quad (4)$$

Equivalently

$$M(x) = \int_0^\infty h(\hat{x}(\tau))d\tau, \quad \hat{x}(0) = x \quad (5)$$

The map $M(\cdot)$ defines implicitly the stable invariant manifold given by the graph $\eta = M(x)$ that is equal to $\lim_{t \rightarrow \infty} \eta(t)$ where the auxiliary variable $\hat{x}$ satisfies

$$\dot{\hat{x}} = f(\hat{x}), \quad \hat{x}(0) = x. \quad (6)$$

**How to compute $M(x)$:** solving the PDE with boundary conditions in (4) or the integral in (5) and the differential equation (6).
The algorithm

Define the global change of co-ordinates

\[ z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(\eta, x) = \begin{bmatrix} \eta - M(x) \\ x \end{bmatrix}, \tag{7} \]

with \( z_1 \in \mathbb{R}, z_2 \in \mathbb{R}^n \). Note that

\[
\frac{\partial T}{\partial (\eta, x)} = \begin{bmatrix} 1 & -\frac{\partial M(x)}{\partial x} \\ 0 & 1 \end{bmatrix}
\]

is non singular for all \((\eta, x) \in \mathbb{R}^{n+1}\) (global diffeomorphism).
The algorithm

In the new co-ordinates the system (2)-(1) is rewritten as

\[
\dot{z}_1 = h(z_2) - \frac{\partial M(z_2)}{\partial z_2} (f(z_2) + g(z_2)u) = -\frac{\partial M(z_2)}{\partial z_2} g(z_2)u, \tag{8}
\]

\[
\dot{z}_2 = f(z_2) + g(z_2)u. \tag{9}
\]

It is possible to analyze the stability properties of the origin \(z = 0\) by the Lyapunov composed function

\[
W(z_1, z_2) = V(z_2) + z_1^2/2.
\]

Note that \(u \equiv 0\) yields

\[
\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) \leq -\rho(\|z_2\|),
\]

proving global stability of \(z = 0\). Furthermore, since \(\rho(\|z_2\|)\) is locally quadratic around the origin, we obtain also that \(z_2 \in L_2\).
The algorithm

To state asymptotic stability of the origin a possible control law $u$ can be retrieved by noting that

$$
\dot{W} = \frac{\partial V(z_2)}{\partial z_2} (f(z_2) + g(z_2)u) - z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)u,
$$

then a possible selection is

$$
u = - \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2),$$
Theorem (1 (Single-Step Integrator Forwarding))

Consider the system (2)-(1), let Assumption 1 hold and suppose that the mapping $M(x)$ is known. Then the control law

$$u(\eta, x) = - \left( \frac{\partial V(x)}{\partial x} - (\eta - M(x)) \frac{\partial M(x)}{\partial x} \right) g(x)$$

renders the origin of the system GAS+LES.

Proof: As first, perform the change of co-ordinates $z = T(\eta, x)$ proposed in (7) that transforms (2)-(1) into (8)-(9).

Consider the Lyapunov function $W(z_1, z_2) = V(z_2) + z_1^2 / 2$, then

$$\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) + \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) u,$$

$$\leq -\rho(\|z_2\|) - \left( \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \right)^2$$


The algorithm: proof...

Then

\[ \dot{W} \leq -\rho(\|z_2\|) - \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 + 2 \frac{\partial V(z_2)}{\partial z_2} g(z_2) z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) + \]

\[ - \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2, \]

\[ \leq -\rho(\|z_2\|) - \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 + \sigma \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 + \]

\[ \frac{1}{\sigma} \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 - \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2, \quad \text{with } \sigma > 0, \]

\[ \leq -\rho(\|z_2\|) - (1 - \sigma) \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 - \left( 1 - \frac{1}{\sigma} \right) \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2, \]

\[ \leq -\rho(\|z_2\|) + \varepsilon \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 - \frac{\varepsilon}{1 + \varepsilon} \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2, \quad (\sigma = 1 + \varepsilon > 1) \]
The algorithm: proof...

Since $V(z_2)$ is locally quadratic around the origin $z_2 = 0$, $g(\cdot)$ is continuous and $\varepsilon > 0$ can be taken arbitrarily small, then there exists a locally quadratic $\tilde{\rho} \in \mathcal{K}$ such that

$$-\rho(\|z_2\|) + \varepsilon \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 \leq -\tilde{\rho}(\|z_2\|)$$

(13)

holds around the origin $z_2 = 0$ yielding

$$\dot{W} \leq -\tilde{\rho}(\|z_2\|) - \frac{\varepsilon}{1 + \varepsilon} \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2.$$
The algorithm: proof...

Note that a global diffeomorphism does not alter the stabilizability nor controllability properties of the Jacobian linearization around the origin, then since in the new co-ordinates the system is

\[
\begin{align*}
\dot{z}_1 &= -\frac{\partial M(z_2)}{\partial z_2} g(z_2) u, \\
\dot{z}_2 &= f(z_2) + g(z_2) u.
\end{align*}
\]

it has to hold that

\[
\left. \frac{\partial M(z_2)}{\partial z_2} \right|_{z_2=0} g(0) \neq 0. \tag{14}
\]

The proof is concluded exploiting La Salle’s Invariance Principle yielding the global asymptotic stability of the origin \( z = 0 \) and then of \( (\eta, x) = (0, 0) \).

Property (14) yields local exponential stability around the origin since

\[
\dot{W} \leq -\bar{\rho}(\|z_2\|) - z_1^2 \frac{\varepsilon}{1 + \varepsilon} \left( \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2.
\]
Why local exponential stability is important?
If $x$ exponentially goes to zero around the origin, the regularity of $h(x)$ (around the origin) allows to conclude that the integral

$$\int_0^\infty h(\hat{x}(t)) \, dt \quad (15)$$

which defines the map $M(x)$, is finite for any value of $x$ (initial condition of $\hat{x} = f(\hat{x})$).

However, this requirement is only a sufficient condition to state the boundedness of the above integral (and the existence of such $M(x)$...).

There is no necessity of the subsystem $x$ to be LES if it is known that (15) is bounded. Nevertheless, in case of recursive Integrator Forwarding, boundedness of (15) have to be satisfied at each step.
The algorithm: single-step example

Consider the system

\[
\dot{x}_1 = \sin(x_2), \quad (16)
\]
\[
\dot{x}_2 = x_1 + u, \quad (17)
\]

and note that it is not in strict-feedforward form as (2)-(1). However, a preliminary control

\[
u = -x_1 - x_2 + u_1
\]
yields

\[
\dot{x}_1 = \sin(x_2), \quad (18)
\]
\[
\dot{x}_2 = -x_2 + u_1, \quad (19)
\]

which is in strict-feedforward form and the Lyapunov function yielding LES (GES) of the \(x_2\) subsystem can be taken as \(V(x_2) = x_2^2/2\). To just apply the formula (22) of the single-step Integrator Forwarding, then set \(\eta = x_1\) and \(x = x_2\), hence

\[
u_1 = -\left(x_2 - (x_1 - M(x_2)) \frac{\partial M(x)}{\partial x}\right).
\]
The algorithm: single-step example

To evaluate the map $M(x)$ we can proceed solving the PDE

\[ i) \quad h(x_2) = \frac{\partial M(x_2)}{\partial x_2} f(x_2), \quad M(0) = 0. \]

or the integral

\[ ii) \quad M(x_2) = \int_0^\infty h(\hat{x}(\tau)) d\tau, \quad \dot{\hat{x}} = -\hat{x}, \quad \hat{x}(0) = x_2. \]

As first, consider $i)$ that rewrites as

\[ \sin(x_2) = -\frac{\partial M(x_2)}{\partial x_2} x_2, \quad M(0) = 0, \]

the solution of which is obtained (via symbolic solver) and is

\[ M(x_2) = -\int_0^{x_2} \frac{\sin(t)}{t} dt. \]
The algorithm: single-step example

Evaluating the integral \( ii \), that is rewritten as

\[
M(x_2) = \int_0^\infty \sin(x_2 e^{-t}) \, dt \quad \Leftarrow \quad \hat{x}(t) = x_2 e^{-t},
\]

whose solution lead to

\[
M(x_2) = -\int_0^{x_2 e^{-t}} \frac{\sin(\tau)}{\tau} \, d\tau \bigg|_{t=\infty}^{t=0} = -\int_0^{x_2} \frac{\sin(t)}{t} \, dt.
\]

Then, since

\[
\frac{\partial M(x_2)}{\partial x_2} = -\frac{\sin(x_2)}{x_2},
\]

the final control law \( u_1 \) is

\[
u_1(x_1, x_2) = -\left( x_2 + \left( x_1 + \int_0^{x_2} \frac{\sin(s)}{s} \, ds \right) \frac{\sin(x_2)}{x_2} \right).
\]
The algorithm: single-step example

An apparently “trivial” system

\[ \dot{x}_1 = \sin(x_2), \quad (20) \]
\[ \dot{x}_2 = x_1 + u, \quad (21) \]

has got a non trivial (even for computation) control

\[ u(x_1, x_2) = -x_1 - x_2 - \left( x_2 + \left( x_1 + \int_0^{x_2} \frac{\sin(s)}{s} \, ds \right) \frac{\sin(x_2)}{x_2} \right), \]

yielding GAS+LES of the origin.

What if we now try to analyze the stability property of the origin?

High gain control?
Suggestion: tray with \( u = -x_1 - k^2 x_1 - k x_2 \) and analyze the stability property of the origin with the function \( V(X) = X' P X \), with \( X = [x_1, x_2]' \) and

\[ P = \begin{bmatrix} k^2 & k/2 \\ k/2 & 1 \end{bmatrix}. \]
Corollary (Saturated control)

In place of the control law (22) in Theorem 1, it is possible to consider the saturated control law

\[ u(\eta, x) = -\sigma \left( \left( \frac{\partial V(x)}{\partial x} - (\eta - M(x)) \frac{\partial M(x)}{\partial x} \right) g(x) \right), \]  

(22)

where the nonlinear saturation function \( \sigma(\cdot) : \mathbb{R}^p \Rightarrow \mathbb{R}^p \) is continuous and such that

\[ \sigma(s)s > 0, \forall s \neq 0, \quad \text{and} \quad \sigma(s)s = \|s\|^2 \text{ in a neighbor of } s = 0. \]  

(23)

Then, the origin of (2)-(1) is GAS+LES.

Proof:
Directly note that

\[
\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) + \\
- \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \sigma \left( \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \right),
\]
Far from the origin it holds that

$$\dot{W} \leq -\rho(\|z_2\|),$$

yielding convergence to zero of $z_2$, furthermore

$$\frac{\partial V(z_2)}{\partial z_2} \bigg|_{z_2=0} = 0$$

$$\downarrow$$

$$= - \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \sigma \left( \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \right) \bigg|_{z_2=0} =$$

$$= - \left( z_1 \frac{\partial M(z_2)}{\partial z_2} \bigg|_{z_2=0} \right) g(0) \sigma \left( \left( z_1 \frac{\partial M(z_2)}{\partial z_2} \bigg|_{z_2=0} \right) g(0) \right) \leq -\gamma(|z_1|)$$

by Assumption 1 for some class-$\mathcal{K}$ function $\gamma(\cdot)$ which implies (by regularity...) that $z_1$ goes to zero. Then, the trajectories of the $z$-system enters in finite time within a sufficiently small neighbor of the origin such that $\sigma(s)s = \|s\|^2$ and the same arguments of Theorem 1’s proof hold allows to conclude the proof. □
Recursive algorithm: an example

Consider the system in *strict-feedforward form* described by

\[
\begin{align*}
\dot{x}_1 &= x_2 + (x_2 - x_3)^2, \quad (24a) \\
\dot{x}_2 &= x_3, \quad (24b) \\
\dot{x}_3 &= -2x_3 + u. \quad (24c)
\end{align*}
\]

**STEP 1:** Consider the subsystem

\[
\begin{align*}
\dot{x}_2 &= x_3, \quad (25a) \\
\dot{x}_3 &= -2x_3 + u, \quad (25b)
\end{align*}
\]

define \( h(x_3) = x_3 \), \( f(x_3) = -2x_3 \) and find the map \( M(x) \) such that

\[
h(x_3) = \frac{\partial M(x_3)}{\partial x_3} f(x_3) \rightarrow x_3 = -2 \left( \frac{\partial M(x_3)}{\partial x_3} \right) x_3,
\]

and \( M(0) = 0 \).

The solution is \( M(x_3) = -x_3/2 \). Define \( z_3 = x_3 \), \( V(z_3) = z_3^2/2, u = u_2 \), \( z_2 = x_2 - M(x_3) = x_2 + x_3/2 \), then
Recursive algorithm: an example

In the new \([z_2, z_3]\) co-ordinates the dynamics of the \((x_2, x_3)\)-subsystem are

\[
\begin{align*}
\dot{z}_2 &= -\frac{\partial M(z_3)}{\partial z_3} u_2 = \frac{u_2}{2}, \\
\dot{z}_3 &= -2z_3 + u_2,
\end{align*}
\]

(26a)

(26b)

and the control law of Theorem 1 yields

\[
 u_2 = -\left(z_3 + \frac{z_2}{2}\right).
\]

Let’s check which is the derivative of the aggregate Lyapunov function

\[
W_2(z_2, z_3) = \frac{z_3^2}{2} + \frac{z_2^2}{2},
\]

\[
\dot{W} = z_3 (-2z_3 + u_2) + z_2 \frac{u_2}{2},
\]

\[
= -2z_3^2 + (z_3 + \frac{z_2}{2})u_2,
\]

\[
= -z_3^2 - \left(z_3 + \frac{z_2}{2}\right)^2,
\]

yielding \((z_2, z_3) = (0, 0)\) GAS+LES.
**Recursive algorithm: an example**

**STEP 2:** Add a new row at the top of the previous subsystem performing the partial change of co-ordinates

\[
\begin{bmatrix}
z_2 \\
z_3
\end{bmatrix} = \begin{bmatrix}
x_2 + x_3/2 \\
x_3
\end{bmatrix}
\]

and letting

\[
u_2 = -\left(z_3 + \frac{z_2}{2}\right) + u_1,
\]

then

\[
\begin{align*}
\dot{x}_1 &= z_2 - \frac{z_3}{2} + \left(z_2 - \frac{3z_3}{2}\right)^2, \\
\dot{z}_2 &= -\frac{2z_3 + z_2}{4} + u_1, \\
\dot{z}_3 &= -2z_3 - \left(z_3 + \frac{z_2}{2}\right) + u_1.
\end{align*}
\]
Recursive algorithm: an example

Let

\[ h(z_2, z_3) = z_2 - \frac{z_3}{2} + \left( z_2 - \frac{3z_3}{2} \right)^2, \quad f(z_2, z_3) = \begin{bmatrix} \frac{2z_3 + z_2}{4} \\ -3z_3 - \frac{z_2}{2} \end{bmatrix}, \]

and find the map \( M(z_2, z_3) \) such that \( M(0, 0) = 0 \) and

\[ h(z_2, z_3) = \frac{\partial M(z_2, z_3)}{\partial(z_2, z_3)} f(z_2, z_3), \]

\[ \downarrow \]

\[ z_2 - \frac{z_3}{2} + \left( z_2 - \frac{3z_3}{2} \right)^2 = -\frac{\partial M(z_2, z_3)}{\partial z_2} \frac{2z_3 + z_2}{4} - \frac{\partial M(z_2, z_3)}{\partial z_3} \left( 3z_3 + \frac{z_2}{2} \right), \]

for which it is really difficult to find out the solution.... however, since (27b)-(27c) is a linear system when \( u_1 = 0 \), we can try to evaluate the integral form of \( M(z_2, z_3) \).
Recursive algorithm: an example

It holds

\[ f(z_2, z_3) = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} := A z \]

Then

\[ \hat{x}(t) = e^{At} z \rightarrow M(z) = \int_0^\infty h(\hat{x}(t)) dt \]

which is a mess but can be computed in closed form....
Let \( m(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( m(x) = [m_1(x), m_2(x), \ldots, m_n(x)]^\top \)

\[ h(x) - m(x)^\top f(x) = 0, \quad \text{such that} \quad m(0)^\top g(0) \neq 0. \]

Instead of finding \( M(x) \) such that

\[ h(x) = L_f M(x), \]

define the map \( \mathcal{M}(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) as

\[ \mathcal{M}(x, \xi) = \sum_{i=1}^n \int_0^{x_i} m_i(\xi) \bigg|_{\xi_i=s} \ ds. \]

Let \( e = \xi - x \), then

\[ \frac{\partial \mathcal{M}(x, \xi)}{\partial x} = m(x)^\top + e^\top \Delta(x, e), \]

\[ e^\top \Delta(x, e) = \frac{\partial \mathcal{M}(x, \xi)}{\partial x} - m(x)^\top = [m_1(x_1, \xi_2, \ldots, \xi_n) - m_1(x_1, x_2, \ldots, x_n), \ldots], \]

\[ = \left[ \sum_{j=1}^n e_j \delta_{1j}(x, e), \sum_{j=1}^n e_j \delta_{2j}(x, e), \ldots, \sum_{j=1}^n e_j \delta_{nj}(x, e) \right], \]

\( \delta_{ij}(\cdot) : \mathbb{R}^{2n} \rightarrow \mathbb{R}, \delta_{ii}(\cdot) \equiv 0. \)
Necessary assumption

Note that

\[
\frac{\partial M(x, 0)}{\partial x} \bigg|_{x=0} g(0) = m(0)^\top g(0) \neq 0.
\]

Let \( z = y - M(x, \xi) \), \( y = \eta \)

\[
\dot{z} = h(x) - \frac{\partial M(x, \xi)}{\partial x} (f(x) + g(x)u) - \frac{\partial M(x, \xi)}{\partial \xi} \dot{\xi},
\]

\[
= -e^\top \Delta(x, e) f(x) - \frac{\partial M(x, \xi)}{\partial \xi} \dot{\xi} - \frac{\partial M(x, \xi)}{\partial x} g(x)u.
\]

The next Assumption is instrumental to prove the main theorems.

**Assumption 2:** There exist a positive definite function \( L(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) and a function \( \gamma(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0} \) such that

i) \[
\frac{\partial L(V)}{\partial V} \frac{\partial V(x)}{\partial x} f(x) + \frac{||f(x)||^2}{2\gamma(x)} \leq -\rho(||x||),
\]

ii) \[
\frac{\partial L(V)}{\partial V} \frac{\partial V(x)}{\partial x} f(x) + \frac{||x||^2}{2\gamma(x)} \leq -\rho(||x||),
\]

for some non-decreasing and locally quadratic at the origin function \( \rho(\cdot) \).
The main Theorem

**Theorem 1 (only dynamic scaling):** Consider the system (1)-(2) and assume Assumptions 1 and 2.i hold. Define the change of coordinates 

\[ [z, x] = [y - M(x, \xi), x] \]

Select \( \dot{\xi} = 0 \) and \( \xi(0) = 0 \) and

\[ \dot{r} = \gamma(x) ||x^\top \Delta(x, -x)||^2 - \frac{r^2 - 1}{1 + z^2} \rho(||x||), \]

(28)

with \( r(0) \geq 1 \), \( \rho(\cdot) \) and \( \gamma(x) \) as in the Assumption 2.i, and the control law

\[ u = - \left( \frac{\partial L}{\partial V} \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x, 0)}{\partial x} \right) g(x)v, \]

(29)

with \( v > 0 \). Then the origin \((x, z) = (0, 0)\) of the closed-loop system is globally asymptotically stable, \((x, u) \in L^2\) and \( r \in L_\infty \). Moreover, if \( L(\cdot) \) is locally quadratic, the origin is locally exponentially stable.
To avoid burden of notation we assume that $V(x)$ in Assumption 1 satisfies also Assumption 2.i with $L(V) = V$ for some $\gamma(x)$. To analyse the stability property of the origin $(x, z) = (0, 0)$ of the closed-loop system we select the composite Lyapunov function

$$W(x, z, \xi) = V(x) + \frac{z^2}{2r},$$

through the dynamic scaling\textsuperscript{1} $r$, time derivative along the system trajectories given by

$$\dot{W} = \left( \frac{\partial V}{\partial x} + \frac{z}{r} x^\top \Delta(x, -x) \right) f(x) + \left( \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x, 0)}{\partial x} \right) g(x) u - \frac{z^2 \dot{r}}{2r^2}.$$  \hfill (30)

\textsuperscript{1}Note that the second term in the rhs of (28) avoids drifting of $r$ in case of measurement noise and, with $r(0) \geq 1$, yields $r \geq 1$. 

---
Proof of Theorem 1 (cont’d)

Using Young’s inequality as

\[
\frac{z}{r} e^\top \Delta(x, -x) f(x) \leq \frac{1}{2} \left( \frac{\gamma(x) z^2}{r^2} \|x^\top \Delta(x, -x)\|^2 + \frac{\|f(x)\|^2}{\gamma(x)} \right),
\]

we have

\[
\dot{W} \leq \frac{\partial V}{\partial x} f(x) + \frac{\|f(x)\|^2}{2\gamma(x)} + \frac{z^2}{2r^2} \left( \gamma(x) \|x^\top \Delta(x, -x)\|^2 - \dot{r} \right) + \left( \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x, 0)}{\partial x} \right) g(x) u.
\]

The choice (28) and (29) yield

\[
\dot{W} \leq -\frac{\rho(\|x\|)}{2} - \left( \left( \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x, 0)}{\partial x} \right) g(x) \right)^2 v,
\]

proving that \( \rho(\|x\|) \in \mathcal{L}_1 \) and \( z/r \in \mathcal{L}_\infty \) (with \( r \geq 1 \)), and \( u \in \mathcal{L}^2 \).
Proof of Theorem 1 (cont’d)

The boundedness of the scaling factor $r$ follows by

$$
\dot{r} = \gamma(x)\|x^\top \Delta(x, -x)\|^2 - \frac{r^2 - 1}{1 + z^2 \rho(\|x\|)},
$$

$$
\leq \|x\|^2 \gamma(x)\|\Delta(x, -x)\|^2,
$$

(32)

by the fact that $\rho(\|x\|)$ is locally quadratic at the origin yields $x \in L^2$, and by the comparison principle[Khalil, Lemma 3.4]. We conclude that the origin of the closed loop system is globally asymptotically stable given that $\dot{W} < 0$ for all $(x, z/r) \neq (0, 0)$, i.e. by boundedness of $r$, $(x, z) \neq (0, 0)$. When $V(\cdot)$ (or in general $L(\cdot)$) is locally quadratic, locally exponential stability of the origin can be proved using recursively the Young’s inequality in (31).
Theorem 2: Consider the system (1)-(2) and assume Assumptions 1 and 2.ii hold. Define the change of coordinates $[z, x] = [y - M(x, \xi), x]$. Select the dynamics of $r$ and $\xi$ as

$$\dot{r} = \gamma(x) \left\| \frac{\partial M(x, \xi)}{\partial \xi} \dot{\xi} \right\|^2 - \frac{r^2 - 1}{1 + z^2} kr, \quad (33)$$

$$\dot{\xi} = -K_e \xi + \dot{x} + K_e x + \frac{z}{r} \Delta(x, e) f(x), \quad (34)$$

with $r(0) \geq 1$, $kr > 0$, $K_e$ positive definite, $\rho(\cdot)$ and $\gamma(x)$ as in Assumption 2.ii and the control law

$$u = - \left( \frac{\partial L}{\partial V} \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x, \xi)}{\partial x} \right) g(x), \quad (35)$$

with $v > 0$. Then the origin $(x, z, e) = (0, 0, 0)$ of the closed-loop system is globally asymptotically stable, $(x, e, u) \in L_2$, and $r \in L_\infty$. Moreover, if $L(\cdot)$ is locally quadratic, the origin is locally exponentially stable.
Proof of Theorem 2

As in the first Theorem’s proof, to avoid burden of notation we assume that $V(x)$ in Assumption 1 satisfies also Assumption 2 with $L(V) = V$ for some $\gamma(x)$. The stability analysis of the origin $(x, z, e) = (0, 0, 0)$ is pursued with the composite Lyapunov function

$$W(x, z, \xi) = V(x) + \frac{z^2}{2r} + \frac{e^\top e}{2},$$

with scaling $r, r \geq 1$ by (28), and with time derivative along the system trajectories

$$\dot{W} = \left( \frac{\partial V}{\partial x} - \frac{z}{r} e^\top \Delta(x, e) \right) f(x) + e^\top \dot{e} - \frac{z^2 \dot{r}}{2r^2} + \left( \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x, \xi)}{\partial x} \right) g(x) u - \frac{z}{r} \frac{\partial M(x, \xi)}{\partial \xi} \dot{\xi}. \quad (36)$$
Proof of Theorem 2 (cont’d)

Using the fact $\mathcal{M}(x, \xi) = x^\top \partial \mathcal{M}(x, \xi) / \partial \xi$ and Young’s inequality as

$$
\frac{z}{r} \frac{\partial \mathcal{M}(x, \xi)}{\partial \xi} \dot{x} = x^\top \frac{z}{r} \frac{\partial \mathcal{M}(x, \xi)}{\partial \xi} \dot{x} \\
\leq \frac{\|x\|^2}{2\gamma(x)} + \frac{z^2}{2r^2} \left\| \frac{\partial \mathcal{M}(x, \xi)}{\partial \xi} \dot{x} \right\|^2 \gamma(x),
$$

yielding

$$
\dot{e} = -K_e e + \frac{z}{r} \Delta(x, e) f(x), \quad (37)
$$

and

$$
\dot{W} \leq \frac{\partial V}{\partial x} f(x) + \frac{\|x\|^2}{2\gamma(x)} + \left( \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial \mathcal{M}(x, \xi)}{\partial x} \right) g(x) u \\
+ \frac{c_1 k_r}{2} - e^\top K_e e, \quad (38)
$$

with $c_1 \leq 1$. 
Proof of Theorem 2 (cont’d)

The selection of \( u \) as in (29), \( K_e \) positive definite, and selecting

\[
k_r = \rho(||x||) + e^\top K_e e + u^2 / v,
\]

yield

\[
\dot{W} \leq -\left( \rho(||x||) + \left( \left( \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial M(x,\xi)}{\partial x} \right) g(x) \right)^2 v + e^\top K_e e \right) / 2
\]

proving that \( \rho(||x||) \in \mathcal{L}_1 \), \( (e,u) \in \mathcal{L}^2 \), and \( z/r \in \mathcal{L}_\infty \) (with \( r \geq 1 \)).

By (37) and \( \rho(||x||) \) locally quadratic at the origin,

\[
||\dot{\xi}||^2 \leq 2 \left( ||\dot{x}||^2 + \frac{z^2}{r^2} ||\Delta(x,e)F(x)||^2 ||x||^2 + ||K_e||^2 ||e||^2 \right)
\]

is integrable, yielding \( r \in \mathcal{L}_\infty \). We conclude that the origin of the closed loop system is globally asymptotically stable given that \( \dot{W} < 0 \) for all \( (x,e,z/r) \neq (0,0,0) \), i.e., by boundedness of \( r \), \( (x,e,z) \neq (0,0,0) \). Local exponential stability of the origin can be proved as in Theorem 1. ■
**Remark 1:** Within the settings of Theorems 1 and 2, there exists a positive definite function $\sigma(\cdot) : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that with

$$u = \frac{z}{r} \frac{\partial M(x, \xi)}{\partial x} g(x) \sigma(x)v,$$

the results of Theorems 1 and 2 hold, respectively.

To meet actuator constraints it is also possible to implement the control law

$$u = \text{sat} \left( \frac{z}{r} \frac{\partial M(x, \xi)}{\partial x} g(x) \sigma(x) \right),$$

(42)
The benchmark example


$$
\begin{align*}
\dot{y} &= x_1 + (x_1 - x_2)^2, \\
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 - 2x_2 + u.
\end{align*}
$$

and the quadratic Lyapunov function $V = x^\top P x$, 

$$
P = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix},
$$

with $\dot{V} = -x_1^2 - x_2^2$. The manifold $M(x)$ is given by

$$
x_1 + (x_1 - x_2)^2 = \frac{\partial M(x)}{\partial x}[x_2, -x_1 - 2x_2]^\top.
$$

We select the approximated (algebraic) solution $m(x)$ as (inconsistent PDE)!

$$
m(x)^\top = [-2 + x_2 - 4x_1, -1 - x_1], \quad (m(0)^\top g(0) \neq 0)
The benchmark example (cont’d)

Then $\mathcal{M}(x, \xi) = (\xi_2 - 2)x_1 - 2x_1^2 - (\xi_1 + 1)x_2$ and

$$\frac{\partial \mathcal{M}(x, \xi)}{\partial x} = [\xi_2 - 2 - 4x_1, -\xi_1 - 1], \quad -x^\top \Delta(x, -x) = [-x_2, x_1].$$

Let $L(V) = (7/5 + \mu)V$ in Assumption 2.i, with $\mu > 0$, and $\gamma(x) = 1$, then $\rho(||x||) = \mu (x_1^2 + x_2^2)$. By Theorem 1, $z = y - (-2x_1 - 2x_1^2 - x_2)$ and

$$\dot{r} = x_1^2 + x_2^2 - \frac{r^2 - 1}{1 + z^2} \rho(||x||),$$

$$u = - \left( (7/5 + \mu) (x_1 + 2x_2) + \frac{z}{r} \right) v.$$ 

Results are compared with the control law

$$u = -(1 + 2x_2 + x_1)(y + 2x_1 + x_2 + (x_1 + x_2)^2/2 + 2x_1^2). \quad (43)$$
The benchmark example (cont’d)

Note that

\[ M(x) - \mathcal{M}(x, 0) = \frac{(x_1 + x_2)^2}{2}. \]

To steer the system closer to the one given by the SJK feedback the following parameters have been chosen: \( v = 10, \mu = 0.5, \) with \( \xi_i(0) = 0 \) and \( r(0) = 1. \)

Simulation results for the benchmark system: control law of Theorem 1 (top), and (43) (bottom).
The benchmark example (cont’d)

Theorem 2: Initial conditions has been selected as $\xi_i(0) = 1$, $\gamma(x) = 0.01$, $K_e = 5I_{2\times2}$, $k_r = \rho(||x||) + e^\top K_e e + u^2/v$ (suggested by the proof of the Theorem),

$$\frac{\partial M(x, \xi)}{\partial \xi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \Delta(x, e) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this case there are not improvements considering the control law of Theorem 2 with respect to the previous one.

Simulation results for the benchmark system: the control law of Theorem 2.