Integrator Forwarding

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References:

- D Carnevale, A Astolfi, “Integrator forwarding without PDEs”, CDC, 2009

ASSN, A.A. 2013-2014
Consider the system in **strict-feedforward form** \((upper\)-triangular structure)\)

\[
\begin{align*}
\dot{\eta} &= h(x), \\
\dot{x} &= f(x) + g(x)u, 
\end{align*}
\]

with \(\eta \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathbb{R}^p\), where \(f(\cdot), h(\cdot) \in C^q\) with \(q \in \mathbb{N}_{\geq 1}\) are zero at zero, \(g(\cdot)\) is continuous and \(g(0) \neq 0\).

**Assumption \((x = 0 \text{ GAS+LES})\)**

The Jacobian linearization of (1)-(2) at \((\eta, x) = (0, 0)\) is stabilizable (controllable). The origin of the system (2), with \(u \equiv 0\), is GAS and LES with Lyapunov function \(V(x)\).

\[\downarrow\]

There exists a class-\(\mathcal{K}\) function \(\rho(\cdot)\) that is locally quadratic around the origin and such that

\[
\dot{V} = \frac{\partial V(x)}{\partial x} f(x) \leq -\rho(\|x\|), \text{ with } u(t) \equiv 0. \tag{3}
\]
Assumption 1 and the properties of $h(x)$, with $u \equiv 0$, are sufficient conditions for the existence of a map $M(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

$$h(x) = \frac{\partial M(x)}{\partial x} f(x), \quad M(0) = 0. \quad (4)$$

Equivalently

$$M(x) = \int_0^\infty h(\hat{x}(\tau)) d\tau, \quad \hat{x}(0) = x \quad (5)$$

The map $M(\cdot)$ defines implicitly the stable invariant manifold given by the graph $\eta = M(x)$ that is equal to $\lim_{t \to \infty} \eta(t)$ where the auxiliary variable $\hat{x}$ satisfies

$$\dot{\hat{x}} = f(\hat{x}), \quad \hat{x}(0) = x. \quad (6)$$

**How to compute $M(x)$**: solving the PDE with boundary conditions in (4) or the integral in (5) and the differential equation (6).
The algorithm

Define the global change of co-ordinates

\[ z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(\eta, x) = \begin{bmatrix} \eta - M(x) \\ x \end{bmatrix}, \]

(7)

with \( z_1 \in \mathbb{R}, \ z_2 \in \mathbb{R}^n \). Note that

\[ \frac{\partial T}{\partial(\eta, x)} = \begin{bmatrix} 1 & -\frac{\partial M(x)}{\partial x} \\ 0 & 1 \end{bmatrix} \]

is non singular for all \((\eta, x) \in \mathbb{R}^{n + 1} \) (global diffeomorphism).
The algorithm

In the new co-ordinates the system (2)-(1) is rewritten as

\[\dot{z}_1 = h(z_2) - \frac{\partial M(z_2)}{\partial z_2}(f(z_2) + g(z_2)u) = -\frac{\partial M(z_2)}{\partial z_2}g(z_2)u, \quad (8)\]

\[\dot{z}_2 = f(z_2) + g(z_2)u. \quad (9)\]

It is possible to analyze the stability properties of the origin \( z = 0 \) by the Lyapunov *composed* function

\[W(z_1, z_2) = V(z_2) + \frac{z_1^2}{2}.\]

Note that \( u \equiv 0 \) yields

\[\dot{W} = \frac{\partial V(z_2)}{\partial z_2}f(z_2) \leq -\rho(\|z_2\|),\]

proving global stability of \( z = 0 \). Furthermore, since \( \rho(\|z_2\|) \) is locally quadratic around the origin, we obtain also that \( z_2 \in \mathcal{L}_2 \).
To state asymptotic stability of the origin a possible control law $u$ can be retrieved by noting that

\[
\dot{W} = \frac{\partial V(z_2)}{\partial z_2} (f(z_2) + g(z_2)u) - z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)u,
\]

\[
= \frac{\partial V(z_2)}{\partial z_2} f(z_2) + \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2)u,
\]

then a possible selection is

\[
u = - \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2),
\]
The algorithm

Theorem (1 (Single-Step Integrator Forwarding))

Consider the system (2)-(1), let Assumption 1 hold and suppose that the mapping $M(x)$ is known. Then the control law

$$u(\eta, x) = -\left( \frac{\partial V(x)}{\partial x} - (\eta - M(x)) \frac{\partial M(x)}{\partial x} \right) g(x)$$

renders the origin of the system GAS+LES.

Proof: As first, perform the change of co-ordinates $z = T(\eta, x)$ proposed in (7) that transforms (2)-(1) into (8)-(9).

Consider the Lyapunov function $W(z_1, z_2) = V(z_2) + z_1^2/2$, then

$$\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) + \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) u,$$

$$\leq -\rho(\|z_2\|) - \left( \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \right)^2$$
The algorithm: proof...

Then

\[ \dot{W} \leq -\rho(\|z_2\|) - \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 + 2 \frac{\partial V(z_2)}{\partial z_2} g(z_2) z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) + \]

\[ - \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 , \]

\[ \leq -\rho(\|z_2\|) - \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 + \sigma \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 + \]

\[ \frac{1}{\sigma} \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 - \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 , \quad \text{with } \sigma > 0, \]

\[ \leq -\rho(\|z_2\|) - (1 - \sigma) \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 - \left( 1 - \frac{1}{\sigma} \right) \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 , \]

\[ \leq -\rho(\|z_2\|) + \varepsilon \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 - \frac{\varepsilon}{1 + \varepsilon} \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 , (\sigma = 1 + \varepsilon > 1) \]
Since $V(z_2)$ is locally quadratic around the origin $z_2 = 0$, $g(\cdot)$ is continuous and $\varepsilon > 0$ can be taken arbitrarily small, then there exists a locally quadratic $\tilde{\rho} \in \mathcal{K}$ such that

$$-\rho(\|z_2\|) + \varepsilon \left( \frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 \leq -\tilde{\rho}(\|z_2\|)$$

(13) holds around the origin $z_2 = 0$ yielding

$$\dot{W} \leq -\tilde{\rho}(\|z_2\|) - \frac{\varepsilon}{1 + \varepsilon} \left( z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2.$$
Note that a global diffeomorphism does not alter the stabilizability nor controllability properties of the Jacobian linearization around the origin, then since in the new co-ordinates the system is

\[
\dot{z}_1 = -\frac{\partial M(z_2)}{\partial z_2} g(z_2) u, \\
\dot{z}_2 = f(z_2) + g(z_2) u.
\]

it has to hold that

\[
\left. \frac{\partial M(z_2)}{\partial z_2} \right|_{z_2=0} g(0) \neq 0.
\] (14)

The proof is concluded exploiting La Salle’s Invariance Principle yielding the global asymptotic stability of the origin \(z = 0\) and then of \((\eta, x) = (0, 0)\). Property (14) yields local exponential stability around the origin since

\[
\dot{W} \leq -\tilde{\rho}(\|z_2\|) - z_1^2 \frac{\varepsilon}{1 + \varepsilon} \left( \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2.
\]
The algorithm

Remark

Why local exponential stability is important?
If $x$ exponentially goes to zero around the origin, the regularity of $h(x)$ (around the origin) allows to conclude that the integral

$$\int_0^\infty h(\hat{x}(t)) \, dt$$ \quad (15)

which defines the map $M(x)$, is finite for any value of $x$ (initial condition of $\hat{x} = f(\hat{x})$).

However, this requirement is only a sufficient condition to state the boundedness of the above integral (and the existence of such $M(x)$...).

There is no necessity of the subsystem $x$ to be LES if it is known that (15) is bounded. Nevertheless, in case of recursive Integrator Forwarding, boundedness of (15) have to be satisfied at each step.
Consider the system

\[
\begin{align*}
\dot{x}_1 &= \sin(x_2), \\
\dot{x}_2 &= x_1 + u,
\end{align*}
\]  

(16) (17)

and note that it is not in strict-feedforward form as (2)-(1). However, a preliminary control

\[u = -x_1 - x_2 + u_1\]

yields

\[
\begin{align*}
\dot{x}_1 &= \sin(x_2), \\
\dot{x}_2 &= -x_2 + u_1,
\end{align*}
\]  

(18) (19)

which is in strict-feedforward form and the Laypunov function yielding LES (GES) of the \(x_2\) subsystem can be taken as \(V(x_2) = \frac{x_2^2}{2}\). To just apply the formula (22) of the single-step Integrator Forwarding, then set \(\eta = x_1\) and \(x = x_2\), hence

\[u_1 = -\left(x_2 - (x_1 - M(x_2)) \frac{\partial M(x)}{\partial x}\right).\]
The algorithm: single-step example

To evaluate the map $M(x)$ we can proceed solving the PDE

\[ i) \quad h(x_2) = \frac{\partial M(x_2)}{\partial x_2} f(x_2), \quad M(0) = 0. \]

or the integral

\[ ii) \quad M(x_2) = \int_{0}^{\infty} h(\hat{x}(\tau))d\tau, \quad \dot{\hat{x}} = -\hat{x}, \quad \hat{x}(0) = x_2. \]

As first, consider \( i) \) that rewrites as

\[ \sin(x_2) = -\frac{\partial M(x_2)}{\partial x_2} x_2, \quad M(0) = 0, \]

the solution of which is obtained (via symbolic solver) and is

\[ M(x_2) = -\int_{0}^{x_2} \frac{\sin(t)}{t} dt. \]
The algorithm: single-step example

Evaluating the integral \( ii \), that is rewritten as

\[
M(x_2) = \int_0^\infty \sin(x_2 e^{-t}) \, dt \quad \Leftarrow \quad \dot{x}(t) = x_2 e^{-t},
\]

whose solution lead to

\[
M(x_2) = -\int_0^{x_2 e^{-t}} \frac{\sin(\tau)}{\tau} \, d\tau \bigg|_{t=0}^{t=\infty} = -\int_0^{x_2} \frac{\sin(t)}{t} \, dt.
\]

Then, since

\[
\frac{\partial M(x_2)}{\partial x_2} = -\frac{\sin(x_2)}{x_2},
\]

the final control law \( u_1 \) is

\[
u_1(x_1, x_2) = -\left( x_2 + \left( x_1 + \int_0^{x_2} \frac{\sin(s)}{s} \, ds \right) \frac{\sin(x_2)}{x_2} \right).
\]
The algorithm: single-step example

An apparently “trivial” system

\[ \dot{x}_1 = \sin(x_2), \quad (20) \]
\[ \dot{x}_2 = x_1 + u, \quad (21) \]

has got a non trivial (even for computation) control

\[ u(x_1, x_2) = -x_1 - x_2 - \left( x_2 + \left( x_1 + \int_0^{x_2} \frac{\sin(s)}{s} ds \right) \frac{\sin(x_2)}{x_2} \right), \]

yielding GAS+LES of the origin.

What if we now try to analyze the stability property of the origin?

High gain control?
Suggestion: tray with \( u = -x_1 - k^2 x_1 - k x_2 \) and analyze the stability property of the origin with the function \( V(X) = X'PX \), with \( X = [x_1, x_2]' \) and

\[ P = \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix}. \]
Corollary (Saturated control)

In place of the control law (22) in Theorem 1, it is possible to consider the saturated control law

\[
u(\eta, x) = -\sigma \left( \left( \frac{\partial V(x)}{\partial x} - (\eta - M(x)) \frac{\partial M(x)}{\partial x} \right) g(x) \right), \tag{22}
\]

where the nonlinear saturation function \( \sigma(\cdot) : \mathbb{R}^p \Rightarrow \mathbb{R}^p \) is continuous and such that

\[
\sigma(s)s > 0, \forall s \neq 0, \quad \text{and } \sigma(s)s = \|s\|^2 \text{ in a neighbor of } s = 0. \tag{23}
\]

Then, the origin of (2)-(1) is GAS+LES.

Proof:

Directly note that

\[
\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) +
\]

\[
- \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \sigma \left( \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \right),
\]

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The algorithm: saturated control - proof

Far from the origin it holds that

\[ \dot{W} \leq -\rho(\|z_2\|), \]

yielding convergence to zero of \( z_2 \), furthermore

\[ \left. \frac{\partial V(z_2)}{\partial z_2} \right|_{z_2=0} = 0 \]

\[ \Downarrow \]

\[ -\left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \sigma \left( \left( \frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2} \right) g(z_2) \right) \bigg|_{z_2=0} = \]

\[ = -\left( z_1 \frac{\partial M(z_2)}{\partial z_2} \bigg|_{z_2=0} \right) g(0) \sigma \left( \left( z_1 \frac{\partial M(z_2)}{\partial z_2} \bigg|_{z_2=0} \right) g(0) \right) \leq -\gamma(|z_1|) \]

by Assumption 1 for some class-\( \mathcal{K} \) function \( \gamma(\cdot) \) which implies (by regularity...) that \( z_1 \) goes to zero. Then, the trajectories of the \( z \)-system enters in finite time within a sufficiently small neighbor of the origin such that \( \sigma(s)s = \|s\|^2 \) and the same arguments of Theorem 1’s proof hold allows to conclude the proof. \( \square \)
Consider the system in *strict-feedforward form* described by

\[
\begin{align*}
\dot{x}_1 &= x_2 + (x_2 - x_3)^2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -2x_3 + u.
\end{align*}
\]  
(24a)
(24b)
(24c)

**STEP 1:** Consider the subsystem

\[
\begin{align*}
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -2x_3 + u,
\end{align*}
\]  
(25a)
(25b)

define \(h(x_3) = x_3\), \(f(x_3) = -2x_3\) and find the map \(M(x)\) such that

\[
\begin{align*}
h(x_3) &= \frac{\partial M(x_3)}{\partial x_3} f(x_3) \rightarrow x_3 = -2 \frac{\partial M(x_3)}{\partial x_3} x_3,
\end{align*}
\]

and \(M(0) = 0\).

The solution is \(M(x_3) = -x_3/2\). Define \(z_3 = x_3\), \(V(z_3) = z_3^2/2\), \(u = u_2\), \(z_2 = x_2 - M(x_3) = x_2 + x_3/2\), then
Recursive algorithm: an example

In the new \([z_2, z_3]\) co-ordinates the dynamics of the \((x_2, x_3)\)-subsystem are

\[
\dot{z}_2 = -\frac{\partial M(z_3)}{\partial z_3} u_2 = \frac{u_2}{2},
\]

\(26a\)

\[
\dot{z}_3 = -2z_3 + u_2,
\]

\(26b\)

and the control law of Theorem 1 yields

\[
u_2 = -\left( z_3 + \frac{z_2}{2} \right).
\]

Let’s check which is the derivative of the aggregate Lyapunov function

\[W_2(z_2, z_3) = \frac{z_2^2}{2} + \frac{z_3^2}{2},\]

\[
\dot{W} = z_3(-2z_3 + u_2) + z_2 \frac{u_2}{2},
\]

\[
= -2z_3^2 + (z_3 + \frac{z_2}{2})u_2,
\]

\[
= -z_3^2 - \left( z_3 + \frac{z_2}{2} \right)^2,
\]

yielding \((z_2, z_3) = (0, 0)\) GAS+LES.
STEEP 2: Add a new row at the top of the previous subsystem performing the partial change of co-ordinates

\[
\begin{bmatrix}
z_2 \\
z_3
\end{bmatrix} = \begin{bmatrix} x_2 + x_3/2 \\ x_3 \end{bmatrix}
\]

and letting

\[u_2 = -\left( z_3 + \frac{z_2}{2} \right) + u_1,\]

then

\[
\dot{x}_1 = z_2 - \frac{z_3}{2} + \left( z_2 - \frac{3z_3}{2} \right)^2, \tag{27a}
\]

\[
\dot{z}_2 = -\frac{2z_3 + z_2}{4} + u_1, \tag{27b}
\]

\[
\dot{z}_3 = -2z_3 - \left( z_3 + \frac{z_2}{2} \right) + u_1. \tag{27c}
\]
Recursive algorithm: an example

Let

\[ h(z_2, z_3) = z_2 - \frac{z_3}{2} + \left( z_2 - \frac{3z_3}{2} \right)^2, \quad f(z_2, z_3) = \begin{bmatrix} -\frac{2z_3 + z_2}{4} \\ -3z_3 - \frac{z_2}{2} \end{bmatrix}, \]

and find the map \( M(z_2, z_3) \) such that \( M(0, 0) = 0 \) and

\[ h(z_2, z_3) = \frac{\partial M(z_2, z_3)}{\partial (z_2, z_3)} f(z_2, z_3), \]

\[ \downarrow \]

\[ z_2 - \frac{z_3}{2} + \left( z_2 - \frac{3z_3}{2} \right)^2 = -\frac{\partial M(z_2, z_3)}{\partial z_2} \frac{2z_3 + z_2}{4} - \frac{\partial M(z_2, z_3)}{\partial z_3} \left( 3z_3 + \frac{z_2}{2} \right), \]

for which it is really difficult to find out the solution.... however, since (27b)-(27c) is a linear system when \( u_1 = 0 \), we can try to evaluate the integral form of \( M(z_2, z_3) \).
Recursive algorithm: an example

It holds

\[ f(z_2, z_3) = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} := A z \]

Then

\[ \hat{x}(t) = e^{At} z \rightarrow M(z) = \int_0^\infty h(\hat{x}(t)) \, dt \]

which is quite nasty but it is known in closed form....