On “uniformity” in definitions of global asymptotic stability for time-varying nonlinear systems

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Abstract

We discuss the lack of “uniformity” in definitions of uniform global asymptotic stability (UGAS) that have been used in various textbooks, monographs, and papers over the years. Sometimes UGAS is taken to be the combination of uniform local stability (ULS) and uniform global attractivity (UGA). Other times it also encompasses uniform global boundedness (UGB). This paper contains an explicit, smooth scalar example that shows that these definitions do not agree in general, even when the right-hand side is locally Lipschitz in the state uniformly in time (and thus bounded in time). We also discuss various notions of global asymptotic stability with relaxed uniformity (with respect to the initial time) requirements for the behavior of the solutions. In particular, we consider class-$\mathcal{K}$ estimates and Lyapunov characterizations.

Keywords: Uniform global asymptotic stability; Uniform boundedness; Nonlinear systems

1. On the lack of “uniformity” in definitions of uniform global asymptotic stability

The literature contains two definitions of uniform global asymptotic stability for the origin of a time-varying nonlinear system $\dot{x} = f(t, x)$. Both definitions include uniform local stability (ULS): for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $t_0 \geq 0$, $|x(t_0)| \leq \delta$ implies $|x(t)| \leq \varepsilon$ for all $t \geq t_0$. Both definitions also contain the notion of uniform global attractivity (UGA): for each pair of strictly positive real numbers $(r, \varepsilon)$ there exists $T > 0$ such that, for any $t_0 \geq 0$, $|x(t_0)| \leq r$ implies $|x(t)| \leq \varepsilon$ for all $t \geq t_0 + T$. These two concepts comprise the definition of uniform global asymptotic stability given in (Bacciotti & Rosier, 2005, Definition 3.6; Hahn, 1967, Definition 36.9\textsuperscript{1}; Khalil, 1996, Definition 3.2; Rouche, Habets, & Laloy, 1977, p. 10; Vidyasagar, 1993, p. 143). In Barbashin and Krasovskii (1954), Massera (1956), Willems (1970, Definition 2.10), Yoshizawa (1975, Definition 9.5) and Khalil (2002, Definition 4.4), the authors add the concept of uniform global boundedness (UGB): for each $r > 0$ there exists $M > 0$ such that, for any $t_0 \geq 0$, $|x(t_0)| \leq r$ implies $|x(t)| \leq M$ for all $t \geq t_0$.

The first purpose of this note is to give an explicit example (see (9)–(10) in Section 3 below) of a time-varying system with right-hand side that is smooth in $(x, t)$ and locally Lipschitz in $x$ uniformly in $t$ that shows

\begin{equation}
\text{ULS}&\text{&UGA}\neq \text{ULS}&\text{&UGA}.
\end{equation}

In particular, the definitions mentioned above are truly different for nonlinear systems. It is worth mentioning that the two definitions agree for linear time-varying systems $\dot{x} = A(t)x$ with $A(\cdot)$ uniformly bounded, which implies that the right-hand side

\textsuperscript{1}In Hahn (1967, equation (36.10)), the author also states that UGAS “could be formulated” as ULS+UGA+UGB, but he doesn’t provide a comparison between the two alternative definitions.
is globally Lipschitz in $x$ uniformly in $t$. It also should be noted that Willems (1970) suggested the relationship (1) without giving a precise example to illustrate this fact, but hinting at a linear example with a right-hand side unbounded in time. The mechanism for generating (1) was outlined simultaneously in Teel and Zaccarian (2006), although the example there is not smooth and is defined less explicitly, in terms of the distance to a set.

2. Definitions of global asymptotic stability with relaxed uniformity requirements

For time-invariant systems with a continuous right-hand side, it is well-known that having local stability (trajectories do not depend on the initial time, so there is no “uniformity” with respect to initial time needed) and global convergence (GC): \( \lim_{t \to \infty} |x(t)| = 0 \) for each trajectory, is equivalent to having ULS and UGA. See, for example, Kurzweil (1956, pp. 68–72). In fact, for continuous, time-invariant systems, having LS and GC is equivalent to the existence of \( x_1, x_2 \in \mathcal{K}_\infty \) (continuous functions from the nonnegative reals to the nonnegative reals that are zero at zero, strictly increasing and unbounded) such that, for all trajectories,

\[
|x(t)| \leq x_1^{-1}(x_2(|x(0)|)|e^{-t(t)}) \quad \forall t \geq 0.
\]  (2)

This fact comes from a combination of Kurzweil (1956) and Sontag (1998, Proposition 7).

For general time-varying systems, having ULS and UGA and UGB is equivalent to a similar bound for all trajectories:

\[
|x(t)| \leq x_1^{-1}(x_2(|x(t_0)|)|e^{-t(t-h_0)}) \quad \forall t \geq t_0 \geq 0.
\]  (3)

This fact results from, for example, Lin, Sontag, and Wang (1996, Proposition 2.5) or Khalil (2002, Lemma 4.5) combined with Sontag (1998, Proposition 7).

Short of the condition (1), one may ask what type of similar bound can be established under the assumptions of ULS and UGA. For continuous time-varying nonlinear systems, already under the conditions ULS and GC, there exist (see Section 4 below) \( x_1, x_2 \in \mathcal{K}_\infty \) such that, for all trajectories,

\[
|x(t)| \leq x_1^{-1}(x_2((1 + t_0)|x(t_0)|)|e^{-t(h_0)}) \quad \forall t \geq t_0 \geq 0.
\]  (4)

The difference between (3) and (4) resides in the argument of the function \( x_2 \). In the former its argument is \( |x(t_0)| \) whereas in the latter its argument is \( (1 + t_0)|x(t_0)| \).

A similar difference appears in the Lyapunov characterizations of ULS and UGA and UGB vs. ULS and GC. For time-varying systems with continuous right-hand side, having ULS and UGA and UGB is equivalent to the existence of \( x_1, x_2 \in \mathcal{K}_\infty \) and a smooth function \( V : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \) such that for all \( (x, t) \in \mathbb{R}^n \times \mathbb{R}_{>0} \),

\[
x_1(|x|) \leq V(x, t) \leq x_2(|x|)
\]  (5)

and

\[
\frac{\partial V(x, t)}{\partial t} + \frac{\partial V(x, t)}{\partial x} f(t, x) \leq -V(x, t).
\]  (6)

This fact was established in Kurzweil (1956) and concurrently, but for systems with locally Lipschitz right-hand sides, in Massera (1956). It was also established a few years earlier in Barbashin and Krasovskii (1954), but under stronger assumptions on the right-hand side of the differential equation.\(^2\) These assumptions imply, in particular, that the right-hand side is locally Lipschitz in \( x \) uniformly in \( t \), and they allow extra conclusions about the derivative of \( V \) with respect to \( x \), namely that it can be bounded independent of \( t \). It is not a coincidence that work aimed at establishing this type of converse Lyapunov theorem adopted the UGAS definition given by the combination of ULS and UGB and UGA.

More recently, converse Lyapunov theorems have appeared giving that, for time-varying systems with locally Lipschitz right-hand side, having ULS and GC is equivalent to the existence of \( x_1, x_2 \in \mathcal{K}_\infty \) and a smooth function \( V : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \) such that for all \( (x, t) \in \mathbb{R}^n \times \mathbb{R}_{>0} \)

\[
x_1(|x|) \leq V(x, t) \leq x_2((1 + t)|x|)
\]  (7)

and

\[
\frac{\partial V(x, t)}{\partial t} + \frac{\partial V(x, t)}{\partial x} f(t, x) \leq -V(x, t).
\]  (8)

This fact follows from Teel and Praly (2000, Corollary 2) once (4) is established, because (4) corresponds to \( \mathcal{K}_\infty \)-stability with respect to the two measures \( (\omega_1, \omega_2) \) with \( \omega_1(x, t) = |x| \) and \( \omega_2(x, t) = (1 + t)|x| \) for the autonomous system \( \dot{x} = f(x, p), p = 1 \). The difference between the two Lyapunov characterizations (5)–(6) and (7)–(8) resides in the argument of the function \( x_2 \) in (5) and (7), respectively. In the first case its argument is \( |x| \) whereas in the second case its argument is \((1 + t)|x|\).

The work in Karafyllis and Tsinias (2003) reports on the existence of a smooth Lyapunov function satisfying (7)–(8) under the assumption of local stability, global attractivity, and global boundedness that are uniform over compact sets of initial states and initial times. In Karafyllis and Tsinias (2003), it is established that these properties also guarantee the bound (4).

Related work can be found in Sontag and Wang (1999) where output stability is considered. Output stability for the system \( \dot{x} = f(x, t), \dot{y} = y \), corresponds to an estimate of the form

\[
|x(t)| \leq x_1^{-1}(x_2((1 + t_0)|x(t_0)|)|e^{-t(t-h_0)}) \quad \forall t \geq t_0 \geq 0.
\]

A Lyapunov characterization of output stability was given in Sontag and Wang (2001). Note that, unlike the other characterizations we have considered, output stability does not insist that \( x = 0 \) is an equilibrium point.

For UGS and UGA, there may exist some useful trajectory estimate or, respectively, Lyapunov characterization, residing
somewhere in between (3) and (4) or, respectively, between (5)–(6) and (7)–(8), that has not been identified yet.

3. A smooth example illustrating the relation (1)

We consider the nonlinear, time-varying system
\[ \dot{x} = f(t, x) := \varphi(t, x)x^3, \]  
where \( \varphi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is defined as
\[ \varphi(t, x) := p_1(t) + p_2(x) + p_3(t, x) \]  
with
\[ p_1(t) := 2 \cos(\pi t / 2), \]
\[ p_2(x) := -3(\tanh(1 - x) + 1), \]
\[ p_3(t, x) := -3(\tanh(x - t) + 1). \]  
\[ (10) \]

Note that \( \varphi(\cdot, \cdot) \) in (10) is smooth with bounded gradient. Moreover, it is not difficult to see that
\[ |\nabla f(t, x)| \leq (6 + \pi)(3x^2 + 2|x|^3) \quad \forall (t, x). \]

It follows from the mean value theorem that \( f \) is locally Lipschitz in \( x \) uniformly in \( t \).

The regions in the \((t, x)\)-plane where \( \varphi \) is greater than one are shown in black in Fig. 1 while the regions where \( \varphi \) is less than \(-1\) are shown in white. The grey regions represent values of \((t, x)\) where \( \varphi(t, x) \) is between \(-1\) and \(1\).

The trajectories starting from different initial conditions and initial times are shown in Fig. 2.

Solutions are unique because \( f \) is smooth. The absence of finite escape times is guaranteed by the fact that, in Fig. 1, the plane is white where \( x \leq 1 \) or \( x \geq t \). Indeed, \( p_1(t) \leq 2 \) for all \( t \) and \( p_2(t) + p_3(t, x) \leq -3 \) if \( x \leq 1 \) or \( x \geq t \). Thus, if \( x(t) \leq 1 \) or \( x(t) \geq t \) then \( \varphi(x(t), t) \leq -1 \), i.e., \((d/dt)|x(t)| \leq -|x(t)|^3 \leq 0 \).

Uniform local stability (see the definition at the beginning of this note) is guaranteed by the fact that, in Fig. 1, the plane is white where \( x \leq 1 \). Indeed, since \( x(t) \leq 1 \) implies \((d/dt)|x(t)| \leq -|x(t)|^3 \leq 0 \), for each \( \varepsilon > 0 \), \( |x(t_0)| \leq \min \{1, \varepsilon\} \) implies \(|x(t)| \leq \varepsilon\).

Uniform global attractivity (see the definition at the beginning of this note) is guaranteed by the fact that, in Fig. 1, the plane is white where \( t \in [2 + 4i, 2 + 4i + \frac{2}{1}] \) for any positive integer \( i \). This follows from the facts that, for all \((t, x)\),
\[ p_2(x) \leq 0 \quad \text{and} \quad p_3(t, x) \leq 0 \quad \text{and, for all} \quad t \in [2 + 4i - \frac{2}{1}, 2 + 4i + \frac{2}{1}] \quad \text{where} \quad i \quad \text{is an arbitrary positive integer}, \]
\[ p_1(t) \geq 2 \cos(\pi t / 4) \geq 1.4 \quad \text{while} \quad p_2(x) + p_3(t, x) \geq 2[-3(\tanh(-2) + 1)] \geq -0.4. \]

Since the width of this interval is one time unit and the time for an absolutely continuous function satisfying \((d/dt)|x(t)| \geq |x(t)|^3\) to escape to infinity from \( x_0 = 3 \) is less than one (in fact less than \( \frac{1}{13} \)), it follows that for each integer \( i \) greater than one, the trajectory starting at \((x(t_0), t_0) = (3, 4i - 0.5)\) satisfies \( x(t) = 4i - 2.5 \) for some \( t > t_0 \) (these nonuniformly bounded trajectories are represented by the bold lines in Fig. 2).

4. ULS & GC imply (4)

Because of Teel and Praly (2000, Proposition 1, Lemma 3), it will be enough to show the following properties:

\[ \text{(1) for each } r > 0 \exists M > 0 \text{ such that} \]
\[ (1 + t_0)|x(t_0)| \leq r, \quad t \geq t_0, \quad t_0 \geq 0 \]
\[ \implies |x(t)| \leq M, \]
\[ (11) \]
\[ \text{(2) for each } r > 0 \text{ and } \varepsilon > 0 \exists T > 0 \text{ such that} \]
\[ (1 + t_0)|x(t_0)| \leq r, \quad t \geq t_0 + T, \quad t_0 \geq 0 \]
\[ \implies |x(t)| \leq \varepsilon. \]
\[ (12) \]

From the ULS property we have that for each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that
\[ t_0 \geq 0, \quad (1 + t_0)|x(t_0)| \leq \delta(\varepsilon) \implies |x(t_0)| \leq \delta(\varepsilon) \]
\[ \implies |x(t)| \leq \varepsilon \quad \forall t \geq t_0. \]
Thus we only need to check condition (11) for \(|x(t_0)| > \delta(1)| because we can take \(M = 1\) otherwise. This implies that we only need to find \(M\) for \(|x(t_0)| \leq r\) and \(t_0 \leq r/\delta(1)\), i.e., from a compact set of initial conditions in state and time. Similarly, we only need to check the condition (12) for \(|x(t_0)| > \delta(e)| because we can take \(T = 0\) otherwise. This implies that we only need to find \(T\) for \(|x(t_0)| \leq r\) and \(t_0 \leq r/\delta(e)\), i.e., from a compact set of initial conditions in state and time.

We establish (12) first. Let \(\mathcal{E} \subseteq \mathbb{R}^n \times \mathbb{R} \geq 0\) be compact and let \(\varepsilon > 0\) be given. We claim that there exists \(T > 0\) such that for each positive integer \(j\) there exists \((x_j(t_0,j), t_0,j) \in \mathcal{E}\), an ensuing solution \(x_j\), and a time \(T_j \geq t_0 + j\) such that \(|x_j(T_j)| > \varepsilon\). Moreover, by ULS,

\[
|x_j(t)| > \varepsilon(t) \quad \forall t \in [t_0,j, t_0,j + j) \subset [t_0, T_j]
\]

(13) (otherwise ULS would imply \(|x_j(T_j)| < \varepsilon\)). Like in the proof of Teel and Praly (2000, Lemma 5), the sequence \(\{x_j\}_{j=1}^{\infty}\) has a subsequence converging to a solution \(x(t)\) starting from some point in \(\mathcal{E}\). Because of (13), \(|x(t)| \geq \delta(t)\) for all \(t \geq t_0\). This contradicts the GC assumption.

Next we establish (11). Recall that we only need to verify that \((1 + |t_0|)|x(t_0)| \leq r\), \(|x(t_0)| \geq \delta(1)|, \(t \geq t_0 \geq 0\) implies \(|x(t)| \leq M\). We first note that the set \(\mathcal{E} := \{(t, x) : (1 + t)|x| \leq r, |x| \geq \delta(1)\}\) is compact. According to the proof of (12), there exists \(T > 0\) such that \(|x(t)| \leq 1\) for all \(t \geq t_0 + T\). It remains to establish the existence of \(M\) such that \((x(t_0), t_0) \in \mathcal{E}\) implies \(|x(t)| \leq M\) for all \(t \in [t_0, t_0+T]\). According to Filippov (1988, Section 1, Theorem 5), the reachable set from a compact set over a compact time interval is compact. Let \(M \geq 1\) denote a bound on the norm of all the elements in this reachable set. Then (11) is satisfied.

References


