A dilated LMI approach to robust performance analysis of linear time-invariant uncertain systems

Yoshio Ebihara*, Tomomichi Hagiwara

Department of Electrical Engineering, Kyoto University, Kyoto-daigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan

Received 18 July 2003; received in revised form 4 April 2005; accepted 8 May 2005
Available online 31 August 2005

Abstract

This paper studies robust performance analysis problems of linear time-invariant systems affected by real parametric uncertainties. In the case where the state-space matrices of the system depend affinely on the uncertain parameters, it is known that recently developed extended or dilated linear matrix inequalities (LMIs) are effective to assess the robust performance in a less conservative fashion. This paper further extends those preceding results and propose a unified way to obtain numerically verifiable dilated LMI conditions even in the case of rational parameter dependence. In particular, it turns out that the proposed dilated LMIs enable us to assess the robust performance via multiaffine parameter-dependent Lyapunov variables so that less conservative analysis results can be achieved. Connections among the proposed conditions and existing results are also discussed concretely. Several existing results can be viewed as particular cases of the proposed conditions.

Keywords: Robust performance analysis; Real parametric uncertainty; Dilated linear matrix inequalities

1. Introduction

Robust performance analysis problems of linear time-invariant (LTI) systems subject to real parametric uncertainties have attracted a great deal of attention. In recent years, linear matrix inequality (LMI) formulations based on the Lyapunov’s stability theorem have been proposed for such analysis problems, and the effectiveness of the LMI-based approaches is now well-recognized. Among them, in the late 1990s, de Oliveira et al. and Peaucelle et al. opened a new horizon by introducing so-called extended or dilated LMI conditions (de Oliveira, Bernussou, & Geromel, 1999; de Oliveira & Skelton, 2001; Peaucelle, Arzelier, Bachelier, & Bernussou, 2000). In these novel LMIs, auxiliary matrix variables are introduced through the reciprocal application of the Elimination Lemma (see, e.g., Skelton, Iwasaki, & Grigoriadis, 1997) so that the Lyapunov variables have no multiplication relation with the state space matrices. In the case where state space matrices of the system depend affinely on the uncertain parameters, this “decoupling” property enables us to assess the robust stability/performance via parameter-dependent Lyapunov variables, which are very promising to alleviate the conservatism stemming from the quadratic stability based analysis conditions (Boyd, Ghaoui, Feron, & Balakrishnan, 1994). The goal of this paper is to further extend those preceding results and provide a unified way to derive numerically verifiable and less conservative diluted LMI conditions even in the case of rational parameter dependence. It turns out that the proposed dilated-LMI-based conditions enable us to assess the robust performance via multiaffine parameter-dependent Lyapunov variables, which is the same class of the parameter-dependent Lyapunov variables employed in de Oliveira et al. (1999), de Oliveira and Skelton (2001) and Peaucelle et al. (2000).

Another contribution of the paper is to give concrete connections among the proposed dilated-LMI-based conditions...
and several existing results (Asai, Hara, & Iwasaki, 2000; Dettori & Scherer, 2000; Feron, Apkarian, & Gahinet, 1996; Fu & Dasgupta, 2000; Gahinet, Apkarian, & Chilali, 1996; Iwasaki & Hara, 1998; Iwasaki & Shibata, 2001). The degree of the conservatism and computational complexity of the proposed conditions vary according to the form of the auxiliary variables to be employed, and several existing conditions can be readily obtained by specializing those variables to specific forms. In particular, if we employ the auxiliary variables depending affinely on the uncertain parameters, it turns out that the proposed robust stability analysis condition coincides with the one by Fu and Dasgupta (2000). Thus we give a new interpretation of this existing condition from the viewpoint of recently developed extended or dilated LMIs (de Oliveira et al., 1999; de Oliveira & Skelton, 2001; Peaucelle et al., 2000), since the condition was originally derived by exploring multipliers in the frequency domain via KYP lemma. In addition, the connection with Fu and Dasgupta (2000) enables us to see that the conditions by Feron et al. (1996) and Gahinet et al. (1996) can be regarded as particular cases of the proposed condition. On the other hand, the results in Asai et al. (2000), Dettori and Scherer (2000), Iwasaki and Hara (1998) and Iwasaki and Shibata (2001) are known to be interpreted within a unified framework of quadratic separation, which was first introduced by Iwasaki and Hara (1998). Although the underlying ideas of these studies are completely different from the one in this paper, we prove that the proposed robust stability analysis condition with parameter-independent auxiliary variables encompasses the one by Dettori and Scherer (2000). Comparisons with other methods in terms of the degree of conservatism and computational complexity will be made via numerical experiments.

We use the following notations. $I_n$ and $0_n$ denote the $n \times n$ identity matrix and zero matrix, respectively. The subscript $n$ is omitted when the size is not relevant or can be determined from the context. For a matrix $A \in \mathbb{R}^{n \times n}$, $A^{-1}$ and $A^T$ are the inverse and transpose of the matrix $A$, respectively, and $A^{-T}$ denotes $(A^{-1})^T$. $\text{He}[A]$ is a shorthand notation for $A + A^T$.

### 2. Preliminary results

The purpose of this section is to lay preliminary results regarding what we call dilated LMIs. To begin with, let us introduce the following lemma.

**Lemma 1.** Let matrices $\mathcal{A} \in \mathbb{R}^{m \times n}, \mathcal{B} \in \mathbb{R}^{m \times l}, \mathcal{C} \in \mathbb{R}^{l \times n}, \mathcal{D} \in \mathbb{R}^{l \times l}$ and a matrix function $\mathcal{M} : \mathbb{R}^{m \times m} \to \mathbb{R}^{n \times n}$ be given. Then, the following two conditions are equivalent:

(i) The matrix $\mathcal{D}$ is invertible and there exists $\mathcal{P} \in \mathbb{R}^{n \times m}$ such that

$$\mathcal{M}(\mathcal{P}) + \text{He}[\mathcal{P}(\mathcal{A} + \mathcal{B}\mathcal{D}^{-1}\mathcal{C})] < 0.$$  \hfill (1)

(ii) There exist matrices $\mathcal{P} \in \mathbb{R}^{n \times m}$, $\mathcal{F}_1 \in \mathbb{R}^{n \times l}$ and $\mathcal{F}_2 \in \mathbb{R}^{l \times l}$ such that

$$\begin{bmatrix} \mathcal{M}(\mathcal{P}) + \text{He}[\mathcal{P}\mathcal{A}] & \mathcal{P}\mathcal{B} \\ \mathcal{B}^T\mathcal{F}_1 & 0 \end{bmatrix} + \text{He}\left\{\begin{bmatrix} \mathcal{F}_1 & \mathcal{C} - \mathcal{D} \end{bmatrix}\right\} < 0.$$  \hfill (2)

Moreover, for every solution $\mathcal{P} = P$ of (1), there exists a sufficiently small $\varepsilon > 0$ such that $(\mathcal{P}, \mathcal{F}_1, \mathcal{F}_2) = (P, PB\mathcal{D}^{-1}, \varepsilon\mathcal{D}^{-1})$ is a solution of (2). Conversely, every matrix $\mathcal{P}$ such that (2) holds for some $\mathcal{F}_1$ and $\mathcal{F}_2$ also satisfies (1).

Although the well-known elimination lemma (Skelton et al., 1997) validates the equivalence of (i) and (ii), we give here an alternative proof so that the latter assertion of Lemma 1 is also verified.

**Proof.** We first show that the condition (i) implies (ii). If (1) holds, then there exists a sufficiently small $\varepsilon > 0$ such that

$$\mathcal{M}(\mathcal{P}) + \text{He}[\mathcal{P}\mathcal{A}] + \frac{\varepsilon}{2}\mathcal{C}^T\mathcal{D}^{-T}\mathcal{D}^{-1}\mathcal{C} < 0.$$  \hfill (2)

Applying the Schur Complement technique (Boyd et al., 1994) to the above inequality, we have

$$\begin{bmatrix} \mathcal{M}(\mathcal{P}) + \text{He}[\mathcal{P}\mathcal{A}] + \frac{\varepsilon}{2}\mathcal{C}^T\mathcal{D}^{-T}\mathcal{D}^{-1}\mathcal{C} & \varepsilon\mathcal{D}^T \mathcal{D}^{-1} \\ \varepsilon\mathcal{D}^{-1}\mathcal{C} & -2\varepsilon I \end{bmatrix} < 0.$$  \hfill (2)

This is nothing but the inequality (2) in the condition (ii) with $\mathcal{F}_1 = \mathcal{B}\mathcal{D}^{-1}$ and $\mathcal{F}_2 = \varepsilon\mathcal{D}^{-1}$.

It remains to show that the condition (ii) implies (i), which is a simple task since (2) yields

$$\begin{bmatrix} I & \mathcal{P} \mathcal{B} \\ \mathcal{B}^T\mathcal{D}^{-1}\mathcal{C} & 0 \end{bmatrix} + \mathcal{M}(\mathcal{P}) + \text{He}[\mathcal{P}\mathcal{A}] < 0.$$  \hfill (2)

The nonsingularity of the matrix $\mathcal{D}$ is ensured by (2) since it implies $\mathcal{F}_2\mathcal{D} + \mathcal{D}^T\mathcal{F}_2 > 0.$  \hfill (2)

In Lemma 1, the matrix inequality (1) can be regarded as an LMI for the analysis and synthesis frequently used in the previous studies (Boyd et al., 1994; Skelton et al., 1997), while (2) is a dilated LMI corresponding to (1). This lemma is a simple but crucial generalization of the results in de Oliveira and Skelton (2001) and Peaucelle et al. (2000). To see this, given a matrix $A \in \mathbb{R}^{n \times n}$ and a matrix variable $P = PT \in \mathbb{R}^{n \times n}$, let us take $\mathcal{M} = 0, \mathcal{A} = 0, \mathcal{B} = I, \mathcal{C} = A, \mathcal{D} = I$ and $\mathcal{P} = P > 0$. Then, the matrix inequality (1) reduces to the Lyapunov inequality $PA + A^TP < 0$ with respect to the Lyapunov variable $P$. On the other hand, the matrix inequality (2) reduces to

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \text{He}\left\{\begin{bmatrix} F_1 & F_2 \end{bmatrix}\begin{bmatrix} A & -I \end{bmatrix}\right\} < 0.$$  \hfill (3)
which gives a dilated LMI with respect to $P$ and the auxiliary variables $F_1$ and $F_2$ corresponding to the Lyapunov inequality. The dilated LMI (3) was introduced in de Oliveira and Skelton (2001) and Peaucelle et al. (2000) aiming at less conservative results for robust stability analysis of polytopic-type uncertain systems. Since the Lyapunov variable $P$ has no cross-product between the matrix $A$, it was shown that the dilated LMI (3) enables us to employ parameter-dependent Lyapunov variables to assess the robust stability. Lemma 1 generalizes this idea, and we see that the dilated LMI (2) successfully eliminates the inverse of the matrix $D$. This is quite appealing in attacking robust stability/performance analysis problems of LTI systems with rational parameter dependence on uncertain parameters, since in such a case we need to deal with an inverse of a matrix containing uncertain parameters (Asai et al., 2000; Fu & Dasgupta, 2000; Iwasaki & Shibata, 2001). It is known that basic tools such as LFT transformations and scaling techniques are effective for dealing with those analysis problems and successful results were obtained with proper refinements (Asai et al., 2000; Fu & Dasgupta, 2000; Iwasaki & Harashima, 1998; Iwasaki & Shibata, 2001). As we will see in the next section, on the other hand, we can overcome the difficulty arising from the matrix inverse in a straightforward way by simply eliminating it by Lemma 1. In addition, Lemma 1 is sufficiently general in the sense that it is applicable to a wide range of performance conditions under both the continuous- and discrete-time setting. Thus, we have established a powerful tool for robust performance analysis of uncertain LTI systems.

3. Robust performance analysis via dilated LMIs

3.1. Dilated-LMI-based conditions for robust stability/performance analysis

Let us consider the uncertain LTI system depicted in Fig. 1. In this figure, the component $G$ is the nominal plant while the matrix $A\delta$ denotes the uncertainties of the system.

Suppose that the nominal part $G$ is a continuous-time LTI system described by

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2q, \\
z &= C_1x + D_1w + D_2q, \\
p &= C_2x + D_{21}w + D_2q,
\end{align*}
\] (4)

where $A \in \mathbb{R}^{n \times n}$, $B_2 \in \mathbb{R}^{n \times l}$, $C_2 \in \mathbb{R}^{l \times n}$ and $D_2 \in \mathbb{R}^{l \times l}$. On the other hand, $A\delta \in \mathbb{R}^{n \times l}$ is a real matrix representing the time-invariant parametric uncertainties of the system and given by

\[
A\delta = \text{diag}(\delta_1I_{k_1}, \ldots, \delta_NI_{k_N}), \quad \delta = (\delta_1, \ldots, \delta_N)^T, \\
\delta \in \mathbb{R}^{N}, \quad |\delta| \leq 1 \quad (i = 1, \ldots, N).
\] (5)

Assuming that the matrix $D_2$ and $A\delta$ satisfy the well-posedness condition on the feedback connection depicted in Fig. 1, the closed-loop system with $q = A\delta p$ can be written as

\[
\begin{align*}
\dot{x} &= (A + B_2(I - A\delta D_2)^{-1}A\delta C_2)x \\
&\quad + (B_1 + B_2(I - A\delta D_2)^{-1}A\delta D_{21})w, \\
z &= (C_1 + D_{12}(I - A\delta D_2)^{-1}A\delta C_2)x \\
&\quad + (D_1 + D_{12}(I - A\delta D_2)^{-1}A\delta D_{21})w \quad (6)
\end{align*}
\]

and we denote the transfer matrix from $w$ to $z$ by $Tzw(s)$. It is known that LTI systems with rational parameter dependence on uncertain parameters can always be written in the form of (6).

We now characterize control performances of the system (6) by dilated LMIs based on Lemma 1.

**Theorem 1 (Stability).** Let us consider the continuous-time uncertain system (6). Then, the following statements are equivalent:

(i) The matrix $I - A\delta D_2$ is invertible and $A + B_2(I - A\delta D_2)^{-1}A\delta C_2$ is stable for all $\delta \in \mathbb{R}^{N}$.

(ii) The matrix $I - A\delta D_2$ is invertible for all $\delta \in \mathbb{R}^{N}$ and there exists $P(\delta) > 0$ such that

\[
\text{He}\{P(\delta)(A + B_2(I - A\delta D_2)^{-1}A\delta C_2)\} < 0. 
\] (7)

(iii) There exist $P(\delta) > 0$ and $F_j(\delta)$ ($j = 1, 2$) such that

\[
\begin{align*}
\text{He}\{P(\delta)A\} &\quad P(\delta)B_2 \\
B_2^TP(\delta) \quad 0
\end{align*}
\]

\[
+ \text{He}\left[ F_1(\delta) \quad F_2(\delta) \right] \left[ A\delta C_2 - (I - A\delta D_2) \right] \quad < 0. \quad (8)
\]

**Proof.** The equivalence of (i) and (ii) is well-known. The equivalence of (ii) and (iii) follows immediately from Lemma 1 with $M = 0$, $P(\delta)$ and with $M = A\delta$, $B_2 = I\delta$, $C = A\delta C_2$ and $D = I - A\delta D_2$. □

**Theorem 2 (H\textsubscript{\infty} performance).** Let us consider the continuous-time uncertain system (6). Then, the following statements are equivalent:

(i) The matrix $I - A\delta D_2$ is invertible and the $H\infty$ cost $\|Tzw(s)\|\infty$ is bounded by $\gamma_{\infty}$ for all $\delta \in \mathbb{R}^{N}$.
(ii) The matrix \( I - A_3 D_2 \) is invertible for all \( \delta \in \Delta \) and there exist \( P_\infty(\delta) > 0 \) such that
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & -\gamma_\infty^2 I
\end{bmatrix} + \text{He}
\begin{bmatrix}
P_\infty(\delta) & 0 & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A & B_1 & 0 \\
C_1 & D_1 & 0
\end{bmatrix}
+ \begin{bmatrix}
B_2 \\
D_{12}
\end{bmatrix}
(I - A_3 D_2)^{-1} A_3 [C_2 \ D_{21} \ 0] < 0. \tag{9}
\]

(iii) There exist \( P_\infty(\delta) > 0 \) and \( F_j(\delta) \) \((j = 1, \ldots, 4)\) such that
\[
\begin{bmatrix}
\text{He}\{P_\infty(\delta)A\} P_\infty(\delta) B_1 \\
B_2^T P_\infty(\delta) & -I & -\gamma_\infty^2 I \\
C_1 & D_1 & 0 \\
B_2^T P_\infty(\delta) & 0 & 0
\end{bmatrix} + \text{He}
\begin{bmatrix}
F_1(\delta) & F_3(\delta) \\
F_2(\delta) & F_4(\delta)
\end{bmatrix}
\begin{bmatrix}
\Delta_\delta C_2 \Delta_\delta D_{21} & 0 \\
0 & -(I - \Delta_\delta D_2)
\end{bmatrix} < 0. \tag{10}
\]

Proof. The equivalence of (i) and (ii) can be seen from the bounded real lemma (Boyd et al., 1994; Skelton et al., 1997). On the other hand, the equivalence of (ii) and (iii) immediately follows from Lemma 1 with \( \mathcal{M} = \begin{bmatrix}
0 & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & -\gamma_\infty^2 I
\end{bmatrix} \),
\[
\mathcal{P} = \begin{bmatrix}
P_\infty(\delta) & 0 \\
0 & 0 \\
B_2 \\
D_{12}
\end{bmatrix}, \quad \text{and with } \mathcal{A} = \begin{bmatrix}
A & B_1 & 0 \\
C_1 & D_1 & 0
\end{bmatrix}, \mathcal{B} = \begin{bmatrix}
0 \\
0 \\
I
\end{bmatrix}, \mathcal{C} = A_3 [C_2 \ D_{21} \ 0] \text{ and } \mathcal{D} = I - A_3 D_2. \quad \Box
\]

We have given dilated LMIs for robust stability analysis (8) and the \( H_\infty \) performance analysis (10). As we have seen in the proofs of Theorems 1 and 2, simple applications of Lemma 1 lead to those dilated LMIs by appropriately selecting the matrix function \( \mathcal{M} \) and the matrices \( \mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \). Furthermore, in both proofs, the matrix \( \mathcal{D} \) corresponds to \( I - A_3 D_2 \) and hence the resulting dilated LMIs successfully eliminate the inverse of \( I - A_3 D_2 \). By virtue of this property, we can derive numerically tractable conditions for robust stability/performance analysis even in the case of rational parameter dependence (see the next subsection). Other control performances such as D-stability (Peaucelle et al., 2000), the \( H_2 \) performance, positive realness (Skelton et al., 1997) and so on can be treated in a similar fashion. It is also straightforward to generalize those results to the analysis of discrete-time systems.

3.2. Robust stability analysis via dilated LMIs

In this subsection, we concentrate our attention on the robust stability analysis problems of the system (6) and explicate the advantages of the proposed dilated LMIs. From Theorem 1, we see that \( A_3 := A + B_2 (I - A_3 D_2)^{-1} A_3 C_2 \) is well-defined and stable for all \( \delta \in \Delta \) if \( I - A_3 D_2 \) is invertible and there exists \( P(\delta) > 0 \) such that (7) holds or iff there exists \( P(\delta) > 0 \) and \( F_j(\delta) \) \((j = 1, 2)\) such that (8) holds. However, as the main difficulty regarding (7) and (8), these conditions have to be verified at infinitely many points over the uncertainty domain \( \delta \).

A usual way to circumvent this difficulty is to look for a parameter-independent Lyapunov variable \( P(\delta) = P \) satisfying (7). The stability test of this kind is known as quadratic stability (Boyd et al., 1994).

Quadratic stability condition (QS) (Boyd et al., 1994)
\[
P > 0, \quad \text{He}\{P(A + B_2 (I - A_3 D_2)^{-1} A_3 C_2)\} < 0 \quad \forall \delta \in \Delta_v. \tag{11}
\]

Here, \( \Delta_v \) is the set of vertices of \( \Delta \) in (5). It should be noted that, in the case of rational parameter dependence (i.e., \( D_2 \neq 0 \)), the implication (11) \( \Rightarrow \) (7) does not hold in general due to the term \( (I - A_3 D_2)^{-1} \). Hence, the above remedy is valid only in the case of affine parameter dependence (i.e., \( D_2 = 0 \)). In stark contrast, the proposed dilated LMI (8) enables us to assess the robust stability via multiaffine parameter-dependent Lyapunov variables, regardless of the form of the dependence on the uncertain parameters. This result can be stated formally as follows.

**Theorem 3** (Dilated LMI condition: D-LMI). The system (6) is well-posed and robustly stable if there exist multiaffine Lyapunov variable \( P_m(\delta) > 0 \) and matrices \( F_1, F_2 \) such that
\[
\begin{bmatrix}
\text{He}\{P_m(\delta)A\} & P_m(\delta) B_2 \\
B_2^T P_m(\delta) & 0
\end{bmatrix} + \text{He}
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\begin{bmatrix}
\Delta_\delta C_2 & -(I - \Delta_\delta D_2)
\end{bmatrix} < 0 \quad \forall \delta \in \Delta_v. \tag{12}
\]

In deriving (12) from (8), we restrict the Lyapunov variable \( P(\delta) \) to be multiaffine and auxiliary variables \( F_1(\delta) \) and \( F_2(\delta) \) to be parameter-independent so that the lefthand side of (8) becomes multiaffine with respect to \( \delta \). If (12) holds, then (8) will be satisfied for all \( \delta \in \Delta \) with \( F_1(\delta) = F_1 \), \( F_2(\delta) = F_2 \) and with \( P(\delta) = P_m(\delta) \). Namely, we can assess the robust stability of \( A_3 \) using multiaffine parameter-dependent Lyapunov variables \( P_m(\delta) \).

From the above discussions, the advantage of D-LMI (12) over QS (11) is apparent if \( D_2 \neq 0 \). Even though both tests are applicable if \( D_2 = 0 \), we see from the latter assertion of Lemma 1 that if (11) holds, then there exists a sufficiently small \( \varepsilon \) such that (12) holds with \( P_m(\delta) = P \), \( F_1 = P B_2 \) and \( F_2 = \varepsilon I \). Thus, we can ensure the advantage of the D-LMI (12) over the QS (11) rigorously in the sense that the D-LMI
(12) always provides no more conservative results than the QS (11).

In Theorem 3, we derive (12) by restricting the auxiliary variables to be parameter-independent. To obtain yet less conservative analysis results, however, it is promising to employ parameter-dependent ones. Indeed, by considering the multiconvexity constraint (Gahinet et al., 1996; Iwasaki & Shibata, 2001), we can employ auxiliary variables that depend affinely on the uncertain parameter \( \delta \).

**Theorem 4 (Dilated LMI condition with multiconvexity: D-LMI-MC).** The system (6) is well-posed and robustly stable if there exist multiaffine Lyapunov variable \( P_m(\delta) > 0 \), matrices \( F_{i1} \) \((i=0, \ldots, N)\) and \( F_{2i} \) \((i=0, \ldots, N)\) such that

\[
\begin{bmatrix}
\text{He}[P_m(\delta)A] & P_m(\delta)B_2 \\
B_2^T P_m(\delta) & 0
\end{bmatrix}
+ \text{He}\left[ \begin{bmatrix} F_{1\delta}(\delta) & 0 \\ 0 & \frac{1}{\delta_0} F_{2\delta}(\delta) \end{bmatrix} \right] \left[ A_\delta C_2 - (I - A_\delta D_2) \right] < 0 \quad \forall \delta \in \delta,
\]

(13a)

\[
\begin{bmatrix}
F_{i1} \\
F_{2i}
\end{bmatrix}
\text{diag}(0, \ldots, 0, k_i, 0, \ldots, 0, 0_n) \\
\times [C_2, D_2] \geq 0 \quad (i = 1, \ldots, N).
\]

(13b)

Here, \( F_{1\delta}(\delta) \) and \( F_{2\delta}(\delta) \) are \( F_{1\delta}(\delta) = F_{10} + \sum_{i=1}^N \delta_i F_{i1} \) and \( F_{2\delta}(\delta) = F_{20} + \sum_{i=1}^N \delta_i F_{2i} \), respectively.

The D-LMI-MC (13) reduces to the D-LMI (12) by letting \( F_{i1} = 0 \) and \( F_{2i} = 0 \) \((i = 1, \ldots, N)\). The freedom of these variables can be used to obtain less conservative analysis results, at the expense of additional computational effort.

4. Connections with the existing results

In this section, we discuss connections among the proposed dilated-LMI-based conditions and several existing results around robust stability analysis.

4.1. Parameter-dependent multiplier (Fu & Desouza, 2000)

We first note that the condition (13) in Theorem 4 essentially coincides with the one by Fu and Desouza (see Theorem 5.1 in Fu & Desouza, 2000). In this preceding study, the condition was derived by applying the KYP lemma to the existence condition of multipliers in the frequency domain. Indeed, the auxiliary variables \( F_{1\delta}(\delta) \) and \( F_{2\delta}(\delta) \) were interpreted as the coefficients of the corresponding multipliers. In stark contrast, we have derived the condition via simple algebraic manipulation related to the dilated LMIs given in Lemma 1. Thus we have given a new interpretation of this condition and clarified in part the connections with recently developed extended or dilated LMIs (de Oliveira et al., 1999; de Oliveira & Skelton, 2001; Peaucelle et al., 2000).

4.2. Generalized Popov criterion (Feron et al., 1996) and AQS test (Gahinet et al., 1996)

Let us consider the uncertain system described by

\[
\dot{x} = \left( A_0 + \sum_{i=1}^N \delta_i A_i \right) x, \quad \delta \in \delta,
\]

(14)

where \( A_i \in \mathbb{R}^{n \times n} \) \((i = 0, \ldots, N)\) are given matrices and \( \delta \) is given by (5). Note that this system can be rewritten in the form of (5) and (6) by letting \( k_i = n \) \((i = 1, \ldots, N)\) and \( A = A_0, B_1 = [A_1 \cdots A_N], C_2 = [I_n \cdots I_n]^T, D_2 = 0_{NN} \).

To assess the robust stability of (14), the following conditions have been proposed.

**Proposition 1 (Generalized Popov criterion (Feron et al., 1996)).** The system (14) is robustly stable if there exist symmetric matrices \( P_i \) \((i = 0, \ldots, N)\), \( S_i \geq 0 \) \((i = 1, \ldots, N)\) and skew symmetric matrices \( T_i \) \((i = 1, \ldots, N)\) such that

\[
P_\delta(\delta) := P_0 + \sum_{i=1}^N \delta_i P_i > 0 \quad \forall \delta \in \delta,
\]

\[
\begin{bmatrix}
\text{He}[P_0 A_0] + C_2^T S_2 C_2 & P_0 B_2 + A_2^T S_2 C_2 + S_2 T_0 \\
B_2^T P_0 + P_2 C_2 A_0 - T_0 C_2 & B_2^T C_2^T P_2 + P_2 C_2 B_2 - S_4
\end{bmatrix}
< 0.
\]

(15)

Here, \( P_\delta = \text{diag}(P_1, \ldots, P_N), T_0 = \text{diag}(T_1, \ldots, T_N) \) and \( S_4 = \text{diag}(S_1, \ldots, S_N) \).

**Proposition 2 (The AQS test (Gahinet et al., 1996)).** The system (14) is robustly stable if there exist symmetric matrices \( P_i \) \((i = 0, \ldots, N)\) such that

\[
P_\delta(\delta) > 0 \quad \forall \delta \in \delta,
\]

\[
\text{He}\left[ P_\delta(\delta) \left( A_0 + \sum_{i=1}^N \delta_i A_i \right) \right] < 0 \quad \forall \delta \in \delta,
\]

\[
P_i A_i + A_i^T P_i \geq 0 \quad (i = 1, \ldots, N).
\]

(16)

(17)

In the following, we prove that those two conditions (16) and (17) can be regarded as particular cases of (13). Similar observations can be found in Fu and Desouza (2000) but we show here explicitly how those conditions (16) and (17) can be obtained by specializing the variables in (13).
Theorem 6. Consider the uncertain system (14) and suppose that the condition (17) in Proposition 2 holds. Then, there always exists a sufficiently small $\varepsilon > 0$ such that the proposed dilated-LMI-based condition (13) for the state-space matrices (15) holds with $P_{\infty}(\delta) = P_{\delta}(\delta)$ and

$$F_{10} = P_{0}B_{2}, \quad F_{ii} = P_{i}B_{2} \quad (i = 1, \ldots, N), \quad F_{20} = \varepsilon I_{NN}, \quad F_{2i} = 0_{NN} \quad (i = 1, \ldots, N).$$

(19)

Proof. Suppose that the second condition in (17) holds. Then, by following similar lines to the proof of Lemma 1, we see there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} \text{He}[P_{\delta}(\delta)A] & P_{\delta}(\delta)B_{2} \\ B_{2}^{T}P_{\delta}(\delta) & 0 \end{bmatrix} + \text{He} \left[ \begin{bmatrix} -A_{0}^{T}C_{2}^{T}P_{d}A_{\delta}C_{2} + \frac{1}{2}C_{2}^{T}S_{d}C_{2} + A_{0}^{T}C_{2}^{T}P_{d}T_{d} + C_{2}^{T}T_{d} \\ -B_{2}^{T}C_{2}^{T}P_{d}A_{\delta}C_{2} \end{bmatrix} \right]< 0 \forall \delta \in \delta_{v}.$$

Hence (13a) holds with $P_{\infty}(\delta) = P_{\delta}(\delta)$ and the auxiliary variables in (19). It is apparent that the condition (13b) holds also since the left-hand side of the inequality in (13b) becomes $\text{diag}(P_{A_{i}} + A_{i}^{T}P_{i}, 0_{NN})$ $(i = 1, \ldots, N)$, respectively, by letting the auxiliary variables as in (19). □

4.3. Analysis conditions based on the quadratic separator

We first introduce the following result that forms an important basis for Asai et al. (2000), Dettori and Scherer (2000), Iwasaki and Hara (1998) and Iwasaki and Shibata (2001).

Proposition 3 (Asai et al., 2000; Dettori and Scherer, 2000; Iwasaki and Hara, 1998). The system (6) is well-posed and robustly stable if there exist a symmetric matrix $\Theta$ and $P > 0$

$$\begin{bmatrix} I \\ A_{\delta} \end{bmatrix}^{T} \Theta \begin{bmatrix} I \\ A_{\delta} \end{bmatrix} \geq 0 \forall \delta \in \delta,$$

$$\begin{bmatrix} \text{He}(PA) & P_{B_{2}} \\ B_{2}^{T}P & 0 \end{bmatrix} + \begin{bmatrix} C_{2} & D_{2} \end{bmatrix}^{T} \Theta \begin{bmatrix} C_{2} & D_{2} \end{bmatrix} < 0.$$  (21)

This LMI-based condition was successfully obtained by Iwasaki and Hara (1998) based on the notion of quadratic separation. However, the resulting LMI (21) is still hard to check numerically since the first condition should be verified at infinitely many points over $\delta$. To circumvent this difficulty, different contributions have been made by Asai et al. (2000), Iwasaki and Hara (1998) and Dettori and Scherer (2000). In the following, we summarize their results and discuss the connections with the proposed dilated-LMI-based conditions.

4.3.1. LFT scaling (Asai et al., 2000) and vertex separator (VS) (Iwasaki & Hara, 1998)

In these studies, the form of the “separator” $\Theta$ is restricted appropriately so that the left-hand side of the first inequality in (21) becomes convex (Asai et al., 2000) and multi-convex (Iwasaki & Hara, 1998) in $\delta$, respectively. Although these studies are rich in their theoretical aspects, the resulting conditions are conservative since they do not go beyond the quadratic stability (Iwasaki & Shibata, 2001). Hence, in the case of affine parameter dependence, the proposed conditions (12) and (13) are always nonconservative than those in Asai et al. (2000) and Iwasaki and Hara (1998). On the other hand, similar assertions do not follow immediately in the case of rational parameter dependence, since in this case it is not yet clear to us whether (12) and (13) encompass the quadratic stability. Nevertheless, the proposed conditions would be promising, in general, since these allow us to assess the robust stability via multiaffine parameter-dependent Lyapunov variables.
4.3.2. Parameter-dependent separator (PDS) (Dettori & Scherer, 2000)

Dettori and Scherer made another contribution by introducing a parameter-dependent separator of the form

$$\Theta = \text{He} \left\{ \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} [A_\delta - I] \right\},$$

where $S_1$ and $S_2$ are variables to be determined. This choice of $\Theta$ renders the left-hand side of the first inequality in (21) identically zero, while the second inequality in (21) becomes affine with respect to $\delta$. Based on these considerations, they successfully derived the following condition for the robust stability analysis of the system (6).

$$\begin{bmatrix} \text{He}(P_m(\delta)A) & P_m(\delta)B_2 \\ B_2^T P_m(\delta) \end{bmatrix} + \text{He} \left\{ \begin{bmatrix} C_2^T S_1 \\ D_2^T S_1 + S_2 \end{bmatrix} [A_\delta C_2 - (I - A_\delta D_2)] \right\} < 0 \quad \forall \delta \in \mathcal{V}.$$  

Note that the above LMI condition allows us to test the robust stability via multiaffine parameter-dependent Lyapunov variables $P_m(\delta) > 0$, which is the same class of the Lyapunov variables in (12) and (13). By comparing the form of the rest of the variables in (23) and (12), we readily obtain the following results.

**Theorem 7.** Consider the uncertain system (6) and suppose that the condition (23) holds. Then, the proposed dilated-LMI-based condition (12) always holds with the same multiaffine Lyapunov variable $P_m(\delta)$ and the auxiliary variables given by $F_1 = C_2^T S_1$ and $F_2 = D_2^T S_1 + S_2$.

**Remark 1.** It should be noted that if we take the matrix $C_2$ to be of full-column rank in (6), the D-LMI (12) and PDS (23) is equivalent in terms of the degree of the conservatism. In other cases, however, the condition D-LMI (12) can be strictly less conservative than PDS (23) as we will see in the numerical experiments in Section 5.

4.3.3. LFT parameter-dependent Lyapunov variables with vertex separator (LFT-VS) (Iwasaki & Shibata, 2001)

In Iwasaki and Shibata (2001), the authors studied existence conditions of the Lyapunov variables depending on the uncertain parameters via LFT form. Indeed, in Theorem 5 of the paper, a necessary and sufficient condition for the existence of “LFT” parameter-dependent Lyapunov variable $P(\delta)$ that satisfies the Lyapunov inequality (7) has been obtained based on the notion of quadratic separator. In addition, by restricting the corresponding separator appropriately, numerically verifiable LMI conditions have been obtained (see Theorem 8 of the paper for details).

As stated in Iwasaki and Shibata (2001), there is no inclusion relationship between the class of multiaffine Lyapunov variables and that of “LFT” Lyapunov variables. Hence, definite results on the comparisons among the proposed conditions (12), (13) and the one in Iwasaki and Shibata (2001) are hardly obtained. In the next section, we compare the degree of conservatism and computational complexity of these conditions via numerical experiments.

### 5. Illustrative examples

Let us consider a mechanical system with two masses and one spring (Gahinet, Nemirovskii, Laub, & Chilali, 1995). The dynamical equation of this system is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \bar{x}_1 \\ 0 & 1 & 0 & 0 & \bar{\dot{x}}_1 \\ 0 & 0 & m_1 & 0 & \bar{\dot{\bar{x}}}_1 \\ 0 & 0 & 0 & m_2 & \bar{\dot{\bar{x}}}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{\dot{x}}_1 \\ \bar{\dot{\bar{x}}}_1 \\ \bar{\dot{\bar{x}}}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u,$$

where the nominal values of the parameters $m_1$, $m_2$ and $k$ are $m_1 = 1.0$, $m_2 = 1.0$ and $k = 1.25$, respectively. We suppose that the following stabilizing controller is applied to this system.

$$u = \frac{2.12s^3 + 5.51s^2 - 6.43s - 1.03}{s^5 + 4.67s^3 + 12.91s^2 + 18.30s + 12.63} \bar{x}_2.$$  

The problem we posed here is to compute the stability margin with respect to the perturbations on the parameters $m_1$, $m_2$ and $k$. Specifically, we consider the following two problems.

**Problem 1** (Affine parameter dependence). Find the largest value of $\gamma$ such that the closed-loop system with $k=1.25+\delta_k$ is robustly stable for all $|\delta_k| < \gamma$.

**Problem 2** (Rational parameter dependence). Find the largest value of $\gamma$ such that the closed-loop system with $m_1 = 1.0 + \delta_{m_1}$, $k = 1.25 + \delta_k$ is robustly stable for all $|\delta_k| < 0.25$ and $|\delta_{m_1}| < \gamma$.

By means of the analysis conditions discussed in Section 4, we carried out bisection search over $\gamma$ and obtained the results in Table 1. In Problem 1, we considered two formulations in terms of Fig. 1 together with (4) and (5) to illustrates the results in Theorems 5–7. Namely, in the first case we take $A_\delta = \delta_k I_k$ and $C_2 = I_k$ as in (15) while in the second case we take $A_\delta = \delta_k$ and $C_2$ of the size $1 \times 8$. The results in Table 1 are consistent with Theorems 5–7. In particular, we see that D-LMI (12) provides a strictly better margin than PDS (23) in the case where $C_2$ does not have full-column rank. It should also be noted that, in these particular examples, D-LMI achieves exactly the same margin as D-LMI-MC with less computational effort.

---

1 We carried out all LMI-related computations by LMI Control Toolbox (Gahinet et al., 1995) on a PC (Athlon 1.7 GHz).
Table 1
Computation results for Problems 1 and 2 (The value $\gamma$ and the average CPU time (s) to determine the feasibility/infeasibility for each fixed $\gamma$ during the bisection.)

<table>
<thead>
<tr>
<th>Method</th>
<th>Problem 1</th>
<th>Problem 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_d = \delta l I_k \in \mathbb{R}^{8 \times 8}$</td>
<td>$A_d = \delta l \in \mathbb{R}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma$</td>
<td>Time (s)</td>
</tr>
<tr>
<td>QS (11)</td>
<td>0.3146</td>
<td>0.03</td>
</tr>
<tr>
<td>GPC (16)</td>
<td>0.8003</td>
<td>0.36</td>
</tr>
<tr>
<td>AQS (17)</td>
<td>0.8003</td>
<td>0.14</td>
</tr>
<tr>
<td>VS (Iwasaki and Hara (1998))</td>
<td>0.3146</td>
<td>0.51</td>
</tr>
<tr>
<td>LFT-VS (Iwasaki &amp; Shibata, 2001)</td>
<td>0.8003</td>
<td>40.41</td>
</tr>
<tr>
<td>PDS (23)</td>
<td>0.8003</td>
<td>0.98</td>
</tr>
<tr>
<td>D-LMI (12) with $P_m(\delta) = P$</td>
<td>0.3146</td>
<td>0.44</td>
</tr>
<tr>
<td>D-LMI (12)</td>
<td>0.8003</td>
<td>0.96</td>
</tr>
<tr>
<td>D-LMI-MC (13)</td>
<td>0.8003</td>
<td>7.93</td>
</tr>
</tbody>
</table>

Table 2
Number of scalar variables in each method

<table>
<thead>
<tr>
<th>Method</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-LMI</td>
<td>$2^{N-1}n(n+1)/l(l+n)$</td>
</tr>
<tr>
<td>D-LMI-MC</td>
<td>$2^{N-1}n(n+1)/(N+1)l(l+n)$</td>
</tr>
<tr>
<td>LFT-VS</td>
<td>$(n+l)(n+l+1)/2+4l(4l+1)/2$</td>
</tr>
</tbody>
</table>

On the other hand, in Problem 2, D-LMI-MC provides better margins than other methods with more computation time. Again, D-LMI achieves the same margin as D-LMI-MC. From this particular numerical example, we see that D-LMI and D-LMI-MC can be strictly less conservative than LFT-VS. Note however that D-LMI and D-LMI-MC may be more conservative than LFT-VS in other problems, since as stated before, the class of Lyapunov variables in LFT-VS is different from that of D-LMI and D-LMI-MC.

Before closing, we discuss the computational complexity of the conditions D-LMI, D-LMI-MC and LFT-VS via the number of LMIs and the number of scalar variables in each LMI condition. Since these conditions assess the robust stability through the vertices of the uncertainty set $\delta$, the number of LMIs grows exponentially with respect to $N$, i.e., the number of uncertain parameters. On the other hand, the number of scalar variables in each condition can be summarized as Table 2, where $n$ and $l$ are the size of $A$ and $A_j$, respectively. It should be noted that the number of variables in LFT-VS depends only on $n$ and $l$ and does not depend on $N$. In view of these facts, D-LMI and D-LMI-MC are computationally more demanding than LFT-VS when $N$ becomes large. On the other hand, it also can be seen from Table 2 that the number of variables in LFT-VS is sensitive to $l$, the size of $A_j$. In Problem 1, LFT-VS requires much more computation time than D-LMI and D-LMI-MC if we take $A_j \in \mathbb{R}^{8 \times 8}$, since in this particular case we have $l = 8$ and thus the number of variables in D-LMI, D-LMI-MC and LFT-VS become 200, 328 and 664, respectively. To reduce the computational burden of D-LMI, D-LMI-MC and LFT-VS, it is preferable to take $A_j$ of smaller size but it is not yet clear whether this is surely suitable for achieving accurate analysis results in general cases.

6. Conclusion

In this paper, we have considered robust performance analysis problems of LTI systems whose state space matrices depend rationally on uncertain parameters. By extending recent results on extended or dilated LMIs, we have shown a unified way to obtain numerically verifiable dilated LMI conditions even in the case of rational parameter dependence (Lemma 1). We have obtained two dilated LMI conditions for robust stability analysis (Theorems 3 and 4), and proved that several existing results can be viewed as particular cases of these conditions (Theorems 5–7). Although the proposed conditions could become numerically demanding especially when dealing with large number of uncertain parameters, numerical experiments in Section 5 have shown that the proposed conditions are indeed promising for achieving less conservative analysis results.

Acknowledgements

This work is supported in part by the Ministry of Education, Culture, Sports, Science and Technology of Japan under Grant-in-Aid for Young Scientists (B), 15760314. The authors are grateful to Prof. Toru Asai and Prof. Gan Chen for the helpful discussions. Suggestions by anonymous reviewers on the connections with Fu and Dasgupta (2000) are also greatly acknowledged.

References


Yoshio Ebihara was born in Fukuoka, Japan, on May 12, 1974. He received the B.E., M.E. and Ph.D. degrees in electrical engineering from Kyoto University, Kyoto, Japan, in 1997, 1999 and 2002, respectively. Since 2002, he has been with the Department of Electrical Engineering, Kyoto University, as a research associate. His research interests include computer-aided control system analysis and design.

Tomonichi Hagiwara was born in Osaka, Japan, on March 28, 1962. He received the B.E., M.E. and Ph.D. degrees in electrical engineering from Kyoto University, Kyoto, Japan, in 1984, 1986 and 1990, respectively. Since 1986 he has been with the Department of Electrical Engineering, Kyoto University, where he is a Professor since 2001. His research interests include sampled-data control, nonlinear/robust stability, two-degree-of-freedom control systems, and dynamical system theory.